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BRANCHING PROCESSES IN SIMPLE RANDOM WALK

MEYER DWASS

ABSTRACT. Let $N(a)$ be the number of overcrossings of height a in a simple random walk. For $p < \frac{1}{2}$, the process $N(0), N(1), \ldots$ is a branching process which eventually becomes extinct. For $\frac{1}{2} < p$, $N(0), N(1), \ldots$ is a stationary process which is a branching process with immigration.

1. Introduction. We show that a certain branching process and a branching process with immigration arise in simple random walk when $p \neq \frac{1}{2}$. By simple random walk we mean the sequence of random variables, $S_0, S_1, \ldots$, $S_0 = 0$, $S_n = X_1 + \cdots + X_n$, $n = 1, 2, \ldots$, where the X_i's are independent and identically distributed;

$$X_i = \begin{cases} 1 & \text{with probability } p, \\ -1 & \text{with probability } 1 - p = q. \end{cases}$$

Definition 1. An *overcrossing* of height a takes place at time n if $S_n = a$, $S_{n-1} = a + 1$, $S_i = a$ for some $i < n$.

Definition 2. $N(a)$ denotes the total number of overcrossings of height a. ($N(a)$ is finite if $p \neq \frac{1}{2}$.)

2. Statement of theorems.

Theorem 1. *For $p < \frac{1}{2}$, $N(0), N(1), \ldots$ evolves as a branching process with*

$$Et^{N(0)} = (1 - p/q)/(1 - pt/q),\text{[1]}$$

$$E(t^{N(a+1)} | N(a) = k) = [q/(1 - pt)]^k, \qquad a, k = 0, 1, \ldots.$$

(In other words, each of the elements of the preceding generation independently gives rise to a random number of new elements, according to the progeny generating function $q/(1 - pt)$.)

Theorem 2. *For $\frac{1}{2} < p$, $N(0), N(1), \ldots$ evolves as a branching process,*

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[1] Without further notice, the dummy variables in all generating functions are assumed to be less than 1 in absolute value.

with immigration, with

$$Et^{N(0)} = (1 - q/p)/(1 - qt/p),$$

$$E(t^{N(a+1)} | N(a) = k) = [p/(1 - qt)]^k [p/(1 - qt)], \qquad a, k = 0, 1, \ldots .$$

(*In other words, each of the elements of the preceding generation independently gives rise to a random number of new elements, according to the progeny generating function* $p/(1 - qt)$; *in addition, in each generation there is an influx of a random number of new individuals by immigration according to the same generating function* $p/(1 - qt)$.)

3. **Proofs.** The proofs of Theorems 1 and 2 will proceed as follows.

Step 1. Define

T = time at which first overcrossing of 0 takes place. (For $p \neq \frac{1}{2}$, $P(T < \infty) < 1$.)

$V(a)$ = total number of overcrossings of a up to time T.

$R_n(i_1, \ldots, i_n)$ = probability of all paths which start at height 1, overcross height 1 i_1 times, overcross height 2 i_2 times, ..., overcross height n i_n times; do not overcross height 0 enroute, and end up eventually at height 1.

$V_n(s_1, \ldots, s_n) = E(s_1^{V(1)} \cdot \ldots \cdot s_n^{V(n)} | T < \infty)$.

Step 2.

(a)
$$P(T < \infty) = \begin{cases} p/q, & p \leq \frac{1}{2}, \\ q/p, & \frac{1}{2} \leq p. \end{cases}$$

(b)
$$E(t^{V(1)} | T < \infty) = \begin{cases} q/(1 - pt), & p \leq \frac{1}{2}, \\ p/(1 - qt), & \frac{1}{2} \leq p. \end{cases}$$

Proof. (a) is a standard fact about simple random walk. Part (b) follows from (a) by the computation,

$$P(V(1) = k,\ T < \infty) = \begin{cases} (p/q)(p \cdot 1)^k q = p^{k+1}, & p \leq \frac{1}{2}, \\ 1 \cdot (p \cdot q/p)^k q = q^{k+1}, & \frac{1}{2} \leq p. \end{cases}$$

Step 3.

$$E(s_2^{V(2)} \cdot \ldots \cdot s_n^{V(n)} | V(1) = k,\ T < \infty) = [V_{n-1}(s_2, \ldots, s_n)]^k, \qquad k = 0, 1, \ldots .$$

Proof. If $k = 0$, then $V(2) = \cdots = V(n) = 0$ and the assertion holds, so assume now that $1 \leq k$. Suppose that $p \leq \frac{1}{2}$. From the definition of R_n,

$$P(V(1) = i_1, \ldots, V(n) = i_n, T < \infty) = (p/q)R_n(i_1, \ldots, i_n)q.$$

For $1 \leq k$, we also have

$$P(V(1) = k, V(2) = i_2, \ldots, V(n) = i_n, T < \infty)$$
$$= \sum (p/q)[pR_{n-1}(j_{12}, \ldots, j_{1n})q] \cdot \cdots \cdot [pR_{n-1}(j_{k2}, \ldots, j_{kn})q]q$$

where the summation is over all vector sums of k $(n-1)$-tuples such that

$$(j_{12}, \ldots, j_{1n}) + \cdots + (j_{k2}, \ldots, j_{kn}) = (i_2, \ldots, i_n).$$

Hence, computing generating functions, we have

$$E(s_1^{V(1)} \cdot \cdots \cdot s_n^{V(n)}; V(1) = k, T < \infty)$$
$$= (p/q)[qE(s_2^{V(1)} \cdot \cdots \cdot s_n^{V(n-1)}; T < \infty)]^k q$$

from which the result follows by dividing through by $P(V(1) = k, T < \infty) = p^{k+1}$. The proof for $\frac{1}{2} \leq p$ is similar and is left to the reader.

Step 4. $V_n(s_1, \ldots, s_n) = V_1(s_1 V_{n-1}(s_2, \ldots, s_n))$.

Proof.

$$E(s_1^{V(1)} \cdot \cdots \cdot s_n^{V(n)} | T < \infty)$$
$$= \sum_k E(s_1^{V(1)} \cdot \cdots \cdot s_n^{V(n)} | V(1) = k, T < \infty) P(V(1) = k | T < \infty)$$
$$= \sum_k s_1^k [V_{n-1}(s_2, \ldots, s_n)]^k P(V(1) = k | T < \infty) \quad \text{(by Step 3)}.$$

This last expression equals the right side of the assertion.

Step 5. Let $Y(0), Y(1), Y(2), \ldots$ be a branching process with $Y(0) = 1$ and $E(t^{Y(n+1)} | Y(n) = k) = F^k(t)$.

Define the joint generating function of $Y(1), \ldots, Y(n)$ to be

$$W_n(s_1, \ldots, s_n) = E(s_1^{Y(1)} \cdot \cdots \cdot s_n^{Y(n)}).$$

Then,

$$W_n(s_1, \ldots, s_n) = F(s_1 W_{n-1}(s_2, \ldots, s_n)).$$

The joint distribution of $Y(1), \ldots, Y(n)$ is uniquely determined by this iterative relationship.

Proof.

$$E(s_1^{Y(1)}\cdot\ldots\cdot s_n^{Y(n)})=\sum_k E(s_1^{Y(1)}\cdot\ldots\cdot s_n^{Y(n)}|Y(1)=k)P(Y(1)=k)$$

$$=\sum_k s_1^k[W_{n-1}(s_2,\ldots,s_n)]^k P(Y(1)=k)=F(s_1W_{n-1}(s_2,\ldots,s_n)).$$

Step 6. Under the condition that $T<\infty$, the process $V(1), V(2),\ldots$ coincides in distribution with a branching process $Y(1), Y(2),\ldots$, where $Y(0)=1$ and

$$E(t^{Y(n+1)}|Y(n)=k)=\begin{cases}[q/(1-pt)]^k, & p\le \tfrac12,\\ [p/(1-pt)]^k, & \tfrac12\le p, \qquad n,\ k=0,1,\ldots.\end{cases}$$

Proof. It follows from Steps 4 and 5 and 2(b) that

$$E(s_1^{V(1)}\cdot\ldots\cdot s_n^{V(n)}|T<\infty)=E(s_1^{Y(1)}\cdot\ldots\cdot s_n^{Y(n)})$$

for all $n=1,2,\ldots$.

Completion of proof of Theorem 1. Suppose that $p<\tfrac12$. If $N(0)=0$ then automatically $N(1)=N(2)=\cdots=0$. Suppose that $N(0)=k>0$. Define $(V_i(1), V_i(2),\ldots, V_i(n))=V_i$ to be the number of overcrossings of heights $1,2,\ldots,n$ between the time of the $(i-1)$st and ith overcrossing of height 0. ($V_1(a)$ is the same as $V(a)$ as defined earlier.) Then:

(a) the random vectors $V_1, V_2,\ldots$ are independent and identically distributed in the sense that

$$P(V_1=v_1,\ldots,V_k=v_k|N(0)=k)$$
$$=P(V_1=v_1|T<\infty)\cdot\ldots\cdot P(V_k=v_k|T<\infty),$$

$P(V_i=v)$ does not depend on i.

(b) $N(a)=V_1(a)+\cdots+V_{N(0)}$ if $N(0)$ is positive.

(c) $P(N(0)=k)=(p/q)^k(1-p/q)$, $k=0,1,\ldots$.

By Step 6, the proof of the theorem is now complete.

Completion of proof of Theorem 2. For $\tfrac12<p$, it is no longer true that if $N(0)=0$ that $N(1), N(2),\ldots$ are also zero. We must now be concerned with the overcrossings of height $a+1$ *after* the last overcrossing of height a. The number of such overcrossings plays the role of the immigration into the population at each generation. An easy computation shows that $P(N(1)=k|N(0)=0, S_0=0)=pq^k$, $k=0,1,\ldots$. For the rest, the proof is similar to that of Theorem 1 and we leave the details to the reader.

4. **Complements.** Suppose that $p < \frac{1}{2}$. A direct calculation shows that

$$P(N(a) = k) = \begin{cases} 1 - (p/q)^{a+1}, \\ (p/q)^a (p/q)^k (1 - p/q), & 0 < k. \end{cases}$$

Hence,

$$G_a(t) = Et^{N(a)} = 1 - (p/q)^{a+1}(1 - t)/(1 - pt/q). \tag{4.1}$$

Let $F(t)$ denote the progeny generating function, $F(t) = q/(1 - pt)$. Since $N(a)$ evolves as a branching process, we must have that

$$G_a(t) = G_0(F^{(a)}(t)), \qquad a = 1, 2, \ldots, \tag{4.2}$$

where $F^{(a)}$ is the a-fold iteration given by $F^{(a)}(t) = F(F^{(a-1)}(t))$, $a = 2, 3, \ldots$. It is easy to verify directly by induction that (4.1) satisfies (4.2).

For $\frac{1}{2} < p$ we have that

$$Et^{N(a)} = (1 - q/p)/(1 - qt/p) = G(t), \qquad a = 1, 2, \ldots.$$

(Since $N(a)$ is a Markov chain it follows that it is a strictly stationary process.) Since $N(0), N(1), \ldots$ evolves as a branching process with immigration, we must have that

$$G(t) = G(F(t))p/(1 - qt) \tag{4.3}$$

reflecting the relationship between $N(a + 1)$ and $N(a)$ stated in Theorem 2, with $F(t) = p/(1 - qt)$. It is easy to verify that (4.3) holds directly.

For $p < \frac{1}{2}$, $N(a)$ must equal 0 for sufficiently large a. This is consistent with the fact that

$$\left.\frac{d}{dt} q/(1 - pt)\right|_{t=1} = p/q < 1.$$

In other words, if the expected number of progeny is less than 1, the branching process becomes extinct with probability 1.

5. **References.** The elementary facts that are needed about random walk and about branching processes can be found in [1].

REFERENCES

1. W. Feller, *An introduction to probability theory and its applications*. Vol. I, 3rd ed., Wiley, New York, 1968. MR 37 #3604.

DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, ILLINOIS 60201