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ON rTH ROOTS IN EIGHTH-GROUPS

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ABSTRACT. Let G be an eighth-group having no relators of the form $R \cong a^t$. If $X^r = a^n$ where a is a generator and X is a word then r divides n and $X = a^{n/r}$.

1. **Introduction.** In 1962 S. Lipschutz published a paper entitled, *On square roots in eighth-groups* in which he proved that if there is no relator $R \cong a^m$ in an eighth-group then the word a^n has no square root for odd exponents and has only the unique square root $a^{n/2}$ for even exponents. In this paper we generalize this result to arbitrary rth roots, namely:

Theorem. *Let G be an eighth-group having no relators of the form $R \cong a^t$. If $X^r = a^n$ where a is a generator then $r|n$ and $X = a^{n/r}$.*

2. **Notation and definitions.** Most of the general notations and definitions of this paper appear in Magnus, Karrass and Solitar [10]. We will use capital letters $A, B, \ldots, V, W$ to denote freely reduced words unless otherwise stated or implied; lower case letters a, b, c, x, y to denote generators, and the letter R with or without subscripts or superscripts to denote a relator.

Notation.

$L(U)$ means the word-length of U.

$A = B$ means that A and B are the same element of the group G.

$A \cong B$ means that A and B are identical words.

$A \approx B$ means that A and B are freely equal.

$A \sim B$ means that A and B are conjugate, that is, there exists T such that $A = TBT^{-1}$.

$A \wedge B$ means that the end of A does not react with the beginning of B. That is, $U \cong AB$ is freely reduced.

$A \vee B$ means that the end of A does react with the beginning of B. That is, $U \cong AB$ is not freely reduced.

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We will use $\overline{U}$ for U^{-1} the inverse of U. U^* denotes a cyclic permutation of U. We say that a word W is less than p/q of a word V, written $W < p/qV$, if $L(S) < p/qL(V)$ for every $S \subset W$ and $S \subset V$. All the standard definitions and basic results concerning eighth-groups can be found in [4], [6] and [7].

Remark. In order to simplify our proofs, we will assume for the rest of this paper that our eighth-groups have only relators whose length is at least 4. The case of relators of length less than 4 can be eliminated without much trouble and will not be considered.

3. Preliminary lemmas and main result.

Lemma 1. *Let G be an eighth-group having no relators of the form $R \cong a^t$. Then $a^n \leq 2/8\, R$ for all integers n.*

Proof. Suppose $a^k \subset a^n$ and $a^k \subset R$. Assume that $k \geq 2$. Then $R' \cong a^k B$ where $B \not\cong a^r$. Consider $R'' \cong a^{k-1}Ba$. If $L(a^{k-1}) \geq L(R)/8$ then $R' \cong R''$. Thus $aB \cong Ba$. But if two elements of a free group commute then they must be powers of some other element of the free group [10]. But the only way for the generator a to be a power of another element is for the element to be a itself. This forces $B \cong a^r$. Hence $R \cong a^{k+r}$— a contradiction. Therefore, $a^{k-1} < 1/8\, R$. Also from this we have that $a < 1/8\, R$. Finally, $a^k < 2/8\, R$. On the other hand if $k = 1$ the lemma is true since $L(R) \geq 4$.

Lemma 2. *Let G be an eighth-group having no relators of the form $R \cong a^t$. If Y is conjugate to a^k then Y must have infinite order.*

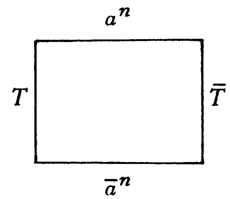
Proof. If Y had finite order, say s, then $(a^k)^s \sim Y^s = 1$. Thus the word a^{ks} would be equal to 1. But an application of Greendlinger's lemma indicates that this power of the generator a would contain a subword greater than $4/8\, R$. Of course this subword itself is a power of a. But by Lemma 1 all powers of a are less than or equal to $2/8\, R$—a contradiction. Hence, Y must have been of infinite order.

Lemma 3. *Let G be an eighth-group. Suppose there is a relator of the form $R \cong XTY\overline{T}$ where X, T, Y are not empty. Then $T < 1/8\, R$.*

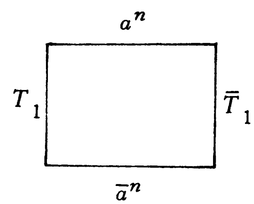
Proof. Let $R' = TY\overline{T}X$ and $\overline{R} = T\overline{Y}\overline{T}\overline{X}$. If $L(T) \geq L(R)/8$ then we have that $R' \cong \overline{R}$. But then $Y\overline{T}X \cong \overline{Y}\overline{T}\overline{X}$. From this we can conclude that $Y \cong \overline{Y}$ in a free group. But there are no nonempty words in a free group with finite order.

Lemma 4. *Let G be an eighth-group having no relators of the form $R \cong a^t$. Suppose $a^n = \overline{T}a^nT$ where T is fully reduced. Then $T \cong a^r$ for some integer r.*

Proof. From the above equation we form the word $a^n\overline{T}\overline{a}^nT = 1$. Next we form the square word:



We would like to apply Greendlinger's lemma to this square word. There could be reaction at the corners of the square word. Suppose $T \cong a^qT_1\overline{a}^p$ where $T_1 \neq$ empty. After the free reduction our new diagram is the following:



Notice that there can be no reaction at the corners of the new square. Also $T_1 \leq 4/8\ R$. In addition $a^n \leq 2/8\ R$ by Lemma 1. Suppose that the circular word is a relator, say $R \cong T_1a^n\overline{T}_1\overline{a}^n$. Now since T_1 and $\overline{T}_1$ both appear in the relator, we have that $T_1 < 1/8\ R$ by Lemma 3. Hence,

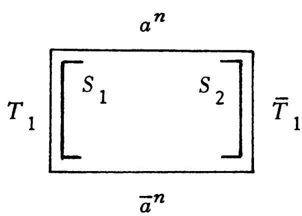
$$L(R) < \frac{1}{8}L(R) + \frac{2}{8}L(R) + \frac{1}{8}L(R) + \frac{2}{8}L(R) = \frac{6}{8}L(R).$$

This contradiction forces us to exclude case (a) of Greendlinger's lemma as a possibility.

Since any 2 adjacent sides of the square word are less than or equal to $6/8\ R$, any piece S whose length is longer than $6/8$ of a relator must cross at least 2 corners. Hence, there cannot be 3 pieces each greater than $6/8$ of a respective relator. This argument eliminates case (c). A similar analysis eliminates cases (d) and (e).

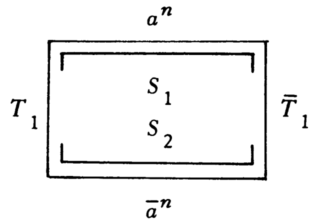
We now concentrate on case (b). Within the square word lie 2 disjoint subwords S_i. Also $L(S_i) > 7L(R_i)/8$. There are only 2 possible ways that this could happen.

(i)



Say $S_1 \cong CT_1D$ and $S_2 \cong E\overline{T}_1F$. Where C and F are subwords of $\overline{a}^n$, and D and E are subwords of a^n. Hence, $C, D, E, F \leq 2/8\, R$ by Lemma 1. Now $L(T_1) \geq L(R_1)/8$ otherwise $L(S_1) \leq 5L(R_1)/8$. But this contradicts $L(S_1) > 7L(R_1)/8$. We consider $R_1 \cong (CT_1D)Z_1$ and $R_2 \cong (E\overline{T}_1F)Z_2$. Let $R_1' \cong T_1DZ_1C$ and $\overline{R}_2' \cong T_1EZ_2F$. Since $L(T_1) \geq L(R_1)/8$ then we have $R_1' \cong R_2'$. This says that $DZ_1C \cong \overline{E}\overline{Z}_2\overline{F}$. But $D \subset a^n$ and $E \subset a^n$. From this we conclude that $a^{-1} \cong a$—a contradiction.

(ii)



By symmetry, we can assume that S_1 contains as many or more symbols on the left as it does on the right. Say $S_1 \cong Da^nK$. Let us say that $T_1 \cong CD$. Then $T_1 \cong \overline{D}\overline{C}$. By choice of C and D, we have that $K \subset \overline{D}$. Thus $\overline{D} \cong KH$. Now

$$R_1 = S_1Z_1 \cong (Da^nK)Z_1 = (\overline{H}\overline{K}a^nK)Z_1.$$

Observe that $(\overline{H}\overline{K}) \subset T_1 \leq 4/8\, R$. $K < 1/8\, R$, and $a^n \leq 2/8\, R$. These facts force $L(R_1) < 8L(R_1)/8$—a contradiction.

If on the other hand S_1 contained more symbols on the right than on the left, then we would have an analogous situation: That is, $S_1 \cong Da^nK$ and $L(K) > L(D)$ where $\overline{T}_1 \cong KC$. Then $T_1 \cong \overline{C}\overline{K}$. But $D \subset T_1$ and $L(D) < L(K)$. Hence, $D \subset \overline{K}$. Therefore, $\overline{K} \cong HD$. Thus $R_1 \cong (Da^nK)Z_1 \cong (Da^n\overline{D}\overline{H})Z_1$. But $\overline{D}\overline{H} \cong K \subset \overline{T}_1 \leq 4/8\, R_1$. Also $a^n \leq 2/8\, R_1$; $D < 1/8\, R$; $Z_1 < 1/8\, R$. These facts force $L(R_1) < 8L(R_1)/8$—a contradiction.

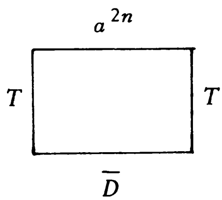
We see from this analysis that no case of Greendlinger's lemma can apply. Therefore, the square word $a^n\overline{T}\overline{a}^nT = 1$ must cyclically reduce to the empty word. This forces $T \cong a^r$.

Lemma 5. *Let G be an eighth-group having no relators of the form $R \cong a^t$. Suppose $a^n \sim X^m$ where X is cyclically fully reduced. Then $m \mid n$ and $X \cong a^{n/m}$.*

Proof. Suppose $a^n \sim X^m$ then $a^{2n} \sim X^{2m}$. Now a^{2n} is cyclically freely reduced. Also since $a^P \leq 2/8\, R$ for any integer P then a^{2n} is cyclically fully reduced. Since X is cyclically freely reduced, X^{2m} is also cyclically freely reduced. We next bend X^{2m} into a circle and fully reduce and freely reduce until the resulting circular word contains no subword greater than $4/8$ of any relator. By this process we obtain a word D_1 which is cyclically fully reduced and also conjugate to X^{2m}. By Greendlinger's basic theorem [4] there is a cyclic permutation of a^{2n} and a cyclic permutation of D_1 (say D) and a conjugating element $T < 1/8\, R$ such that $a^{2n} = TD\overline{T}$.

Of all such pairs (D, a^{2n}) and all such conjugating elements T where D is a cyclic permutation of D_1 and in which $a^{2n} = TD\overline{T}$, we chose the conjugating element T of minimal length.

(1) Suppose T is not empty. We next consider the "square" word $a^{2n}T\overline{D}\overline{T} = 1$:



By our choice of a conjugating element T of minimal length we see that there can be no reaction at the corners of the above square word.

We now apply Greendlinger's lemma to our square word. We will show that none of the 5 cases of that tool can apply.

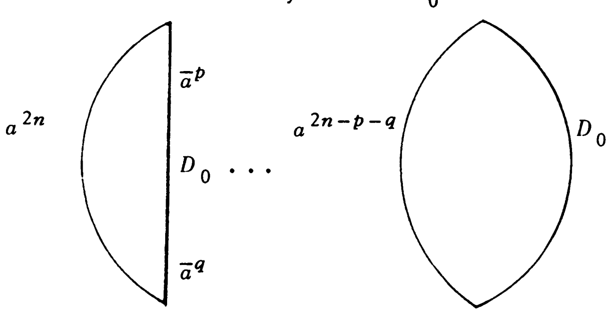
(a) Suppose $R \cong a^{2n}T\overline{D}\overline{T}$ were a relator. But then

$$L(R) < \frac{2}{8}L(R) + \frac{1}{8}L(R) + \frac{4}{8}L(R) + \frac{1}{8}L(R) = \frac{8}{8}L(R)$$

—a contradiction.

(b) By using a minimal conjugating element T, Greendlinger showed [4] that cases (c), (d) and (e) of Greendlinger's lemma could not apply. Also if case (b) applies then $R_1 \cong D_3\overline{T}a^kZ_1$ and $R_2 \cong a^rTD_1Z_2$. But if $L(T) \geq L(R_1)/8$ then one can see that $R_1' \cong R_2'$, consequently, $a \cong \overline{a}$—a contradiction. On the other hand, if $L(T) < L(R_1)/8$ then clearly $L(R_1) < 8L(R_1)/8$ —a contradiction. We see that none of Greendlinger's 5 cases can satisfy our square word. Hence, the case we started with cannot occur.

(2) Suppose T is empty. Then $a^{2n} = D$. Also suppose that $a^{2n}\overline{D}$ does not freely reduce to the empty word. Suppose there is reaction between a^{2n} and $\overline{D}$; reaction between $\overline{D}$ and a^{2n}. Say $\overline{D} \cong \overline{a}^pD_0\overline{a}^q$.



Notice that $a^{2n-p-q} \le 2/8\, R$ and $D_0 \le 4/8\, R$. This circular word is now cyclically reduced and equals 1 but cannot satisfy any of the 5 cases of Greendlinger's lemma since $(a^{2n-p-q})(D_0) \le 6/8\, R$. Hence, the case in which T is empty and $a^{2n}\overline{D}$ does not freely reduce to the empty word cannot happen. Therefore, $a^{2n}\overline{D}$ must freely reduce to the empty word. Finally, we must have that $D \cong a^{2n}$.

These arguments show that if a^{2n} is conjugate to a cyclically fully reduced word then that word must be identically equal to a^{2n}. We end the proof of the theorem by considering 2 cases:

(a) X^2 is cyclically fully reduced;

(b) X^2 is not cyclically fully reduced.

(a) Suppose X^2 is cyclically fully reduced. By Lemma 2, X must have infinite order, since $a^n \sim X^m$. We now use Lemma 4 of Lipschutz [7]. By that paper we can conclude that X^{2m} is cyclically fully reduced. So when we begin to cyclically fully reduce X^{2m} as is the procedure in the use of Greendlinger's basic theorem on the conjugacy problem [4] there can be no reduction. Hence, our "D" is simply a cyclic permutation of X^{2m}. But by the previous arguments, D is identically equal to a^{2n}. Therefore, $X^{2m} \cong a^{2n}$. Thus, $X \cong a^q$. Then $(a^q)^{2m} \cong a^{2n}$. Therefore, $qm = n$. So $m \mid n$ and $X \cong a^{n/m}$.

(b) Suppose X^2 is not cyclically fully reduced. Then by the same reference mentioned in part (a) there exists $R \cong W_1W_2W_1\overline{T}_1$ where W_1W_2 is some cyclic permutation of X and $(T_1W_2)^m$ is cyclically fully reduced. Therefore, when we bend X^{2m} into a circle, we can also write (W_1W_2) around the circle $(2 \cdot m)$ times. Each time we replace $W_1W_2W_1$ by T_1. Therefore, we arrive at $(T_1W_2)^m$ written around the circle. There can be no reduction by the above mentioned paper. Therefore, "D" is just some cyclic permutation of $(T_1W_2)^m$. So we finally have that a^{2n} is conjugate to the cyclically fully reduced word $(T_1W_2)^m$. Hence, by our previous discussion $(T_1W_2)^m \cong a^{2n}$. As a consequence $T_1 \cong a^t$ and $W_2 \cong a^q$. Then $R \cong W_1W_2W_1T_1 \cong W_1a^qW_1\overline{a}^t$. $R' \cong W_1\overline{a}^tW_1a^q$. If $W_1 \ge 1/8\, R$ then $R \cong R'$. Hence, $a^qW_1\overline{a}^t \cong \overline{a}^tW_1a^q$. This means that $a \cong \overline{a}$ which is impossible. But on the other hand $W_1 < 1/8\, R$ then

$$L(R) < \frac{1}{8}L(R) + \frac{2}{8}L(R) + \frac{1}{8}L(R) + \frac{2}{8}L(R) = \frac{6}{8}L(R)$$

—a contradiction. These arguments show that case (b) cannot happen.

Theorem. *Let G be an eighth-group having no relators of the form $R \cong a^t$. If $X^r = a^n$ where a is a generator, then $r \mid n$ and $X = a^{n/r}$.*

Proof. Let X be an arbitrary word and suppose that $X^r = a^n$. By reduction process we can find $X_0 = X$ in which X_0 is fully reduced. Thus $X_0^r = X^r = a^n$. Now X_0 may not be cyclically fully reduced. If $X_0 = (ABC)$ with $L(CA) > L(R)/2$ and $R \cong (CA)\overline{T}_1$. Then $\overline{A}X_0A = \overline{A}(ABC)A = B(CA) = BT_1$. Hence, $X_0 \sim (BT_1)$ and $L(T_1) < L(CA)$. By repetition of this process, if necessary, we can find a cyclically fully reduced word X_1 which is conjugate to X_0. Therefore, $X_0 = TX_1\overline{T}$ where X_1 is cyclically fully reduced. We may also take T to be fully reduced.

Now, $a^n = X_0^r = (TX_1\overline{T})^r = (TX_1^r\overline{T})$. But then $X_1^r \sim a^n$ with X_1 cyclically fully reduced. Hence, by Lemma 5 we conclude that $r \mid n$ and $X_1 \cong a^{n/r}$. Let $k = n/r$. Hence, $X_0 = TX_1\overline{T} = Ta^k\overline{T}$. Now by hypothesis $a^n = X_0^r = (Ta^k\overline{T})^r = Ta^n\overline{T}$. We now use Lemma 4 and conclude that $T \cong a^t$.

Finally, $X = X_0 = TX_1\overline{T} = a^t a^k \overline{a}^t \approx a^k$. Therefore, $X = a^{n/r}$.

BIBLIOGRAPHY

1. M. Dehn, *Über unendliche diskontinuierlicke Gruppen*, Math. Ann. 71 (1911), 116–144.

2. ———, *Transformation der Kurven auf zweiseitigen Flächen*, Math. Ann. 72 (1912), 413–421.

3. M. D. Greendlinger, *Dehn's algorithm for the word problem*, Comm. Pure Appl. Math. 13 (1960), 67–83. MR 23 #A1693.

4. ———, *On Dehn's algorithm for the conjugacy and word problems, with applications*, Comm. Pure Appl. Math. 13 (1960), 641–677. MR 23 #A2327.

5. Seymour Lipschutz, *Elements in S-groups with trivial centralizers*, Comm. Pure Appl. Math. 13 (1960), 679–683. MR 23 #A2455.

6. ———, *On square roots in eighth-groups*, Comm. Pure Appl. Math. 15 (1962), 39–43. MR 29 #141.

7. ———, *An extension of Greendlinger's results on the word problem*, Proc. Amer. Math. Soc. 15 (1964), 37–43. MR 28 #4018.

8. ———, *Powers in eighth-groups*, Proc. Amer. Math. Soc. 16 (1965), 1105–1106. MR 31 #5894.

9. ———, *On the conjugacy problem and Greendlinger's eighth-groups*, Proc. Amer. Math. Soc. 23 (1969), 101–106. MR 40 #4343.

10. W. Magnus, A. Karrass and D. Solitar, *Combinatorial group theory: Presentations of groups in terms of generators and relations*, Pure and Appl. Math., vol. 13, Interscience, New York, 1966. MR 34 #7617.

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