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ALGEBRAS SATISFYING CONGRUENCE RELATIONS

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ABSTRACT. It is shown that the classical nonassociative algebras which have an identity element can be defined in terms of congruence relations modulo the base field.

1. **Introduction.** In an earlier note [2] two of the authors have shown that if an algebra A with identity element 1 over a field F satisfies the property that $(xy)z - x(yz) \in F \cdot 1$ for all x, y, z in A, then A is an associative algebra. Here we consider the similar question for the classical nonassociative algebras. Recall than an alternative algebra is one which satisfies $x^2y - x(xy) = yx^2 - (yx)x = 0$ for all elements x, y and a (linear) Jordan algebra is a commutative algebra over a field of characteristic $\neq 2$ which satisfies $(xy)x^2 - x(yx^2) = 0$ for all elements x, y. In our main results we show that if A is an algebra with identity element 1 over a field F such that $xy - yx \in F \cdot 1$ and $(xy)x^2 - x(yx^2) \in F \cdot 1$ for all elements x and y, then A is a Jordan algebra. Also if the characteristic of F is not 3 and if $x(xy) - x^2y \in F \cdot 1$ and $(yx)x - yx^2 \in F \cdot 1$ for all elements x and y, then A is an alternative algebra. Similar results for strongly alternative algebras, noncommutative Jordan algebras, and power-associative algebras are established.

As usual (x, y, z) denotes $(xy)z - x(yz)$ and $[x, y]$ denotes $xy - yx$. Wherever convenient we will write $a \equiv 0 \bmod F$ instead of $a \in F \cdot 1$. Also, throughout we shall use the term "algebra" to mean a not necessarily associative algebra with identity element 1 over a field F.

Our results depend on the Teichmüller or 5-identity:

$$x(y, z, w) + (x, y, z)w = (xy, z, w) - (x, yz, w) + (x, y, zw)$$

which holds in any nonassociative ring. We also rely heavily on the ability to linearize identities [3], [5] and on the linear independence of various elements of our algebra. Thus, we restrict our attention to algebras over fields.

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2. The alternative case. Recall that an algebra is called flexible if $(x, y, x) = 0$ for all elements x, y.

Lemma 1. *If A is an algebra in which $(x, x^2, x) \equiv 0 \bmod F$ for all x, y in A, then $2(x, x, x) = 0$ for all x in A.*

Proof. The result is trivially true if $x \in F \cdot 1$. Assume now that $x \notin F \cdot 1$. In $(x, x^2, x) \equiv 0 \bmod F$, replace x by $x + 1$ to yield $2(x, x, x) \equiv 0 \bmod F$. Next, linearize the relation $2(x, x, x) \equiv 0 \bmod F$ to get $2(x^2, x, x) + 2(x, x^2, x) + 2(x, x, x^2) \equiv 0 \bmod F$. Thus $2(x^2, x, x) + 2(x, x, x^2) \equiv 0 \bmod F$. By the 5-identity

$$x(x, x, x) + (x, x, x)x = (x^2, x, x) - (x, x^2, x) + (x, x, x^2).$$

Since $2(x, x, x) \in F \cdot 1$ we have $4x(x, x, x) \in F \cdot 1$. But $x \notin F \cdot 1$. Therefore $4(x, x, x) = 0$ and so $2(x, x, x) = 0$.

Lemma 2. *If A is an algebra in which $(x, x, y) \equiv (y, x, x) \equiv 0 \bmod F$ for all x, y in A, then A is flexible.*

Proof. Expand $(x + y, x + y, x) \in F \cdot 1$ to get $(x, y, x) \in F \cdot 1$. Thus Lemma 1 applies.

By the 5-identity we have

$$(x^2, x, y) - (x, x^2, y) + (x, x, xy) - (x, x, x)y = x(x, x, y) \tag{1}$$

and

$$(y, x, x^2) - (y, x^2, x) + (yx, x, x) - y(x, x, x) = (y, x, x)x. \tag{2}$$

Now add equations (1) and (2). Since $(x, y, x) \in F \cdot 1$ and $2(x, x, x) = 0$ it follows that the left side of the resulting equation is in $F \cdot 1$. Therefore $x(x, x, y) + (y, x, x)x \in F \cdot 1 \cap F \cdot x$. Thus we have

$$(x, x, y) + (y, x, x) = 0. \tag{3}$$

From the 5-identity again, we get

$$(x^2, y, x) = x(x, y, x) + (x, x, y)x + (x, xy, x) - (x, x, yx). \tag{4}$$

Therefore $(x^2, y, x) \in F \cdot 1 + F \cdot x$. This gives $(x^2, x, y) \in F \cdot 1 + F \cdot x$ and $(x, x^2, y) \in F \cdot 1 + F \cdot x$. From (1) we can now conclude that $(x, x, x)y \in F \cdot 1 + F \cdot x$ for all x, y in A. Thus $(x, x, x) = 0$ for all x in A. For if $\dim A > 2$, an element y can be chosen such that $y \notin F \cdot 1 + F \cdot x$ whereas if $\dim A \leq 2$, it is automatic that A is associative.

Now if F contains more than two elements, we can linearize $(x, x, x) = 0$ to obtain $(x, x, y) + (y, x, x) + (x, y, x) = 0$ and from (3) it follows that A is flexible. If F contains only two elements, we imbed F in a larger field $\overline{F}$ and consider the scalar extension $\overline{A} = A \otimes_F \overline{F}$. Since $(x, x, y) \in F \cdot 1$ and $(y, x, x) \in F \cdot 1$ for all x, y in F, it is immediate that $(\overline{x}, \overline{x}, \overline{y}) \in \overline{F} \cdot 1$ for all $\overline{x}, \overline{y}$ in $\overline{A}$. Consequently $\overline{A}$ is flexible and thus A is flexible also. □

San Soucie [6] has defined a strongly right alternative algebra to be a right alternative algebra ($(y, x, x) = 0$) satisfying the identity $((xy)z)y = x((yz)y)$. Thedy [9] has shown that for a right alternative algebra this is equivalent to $(y, x, x^2) = 0$ in all extensions. We now prove

Lemma 3. *If A is an algebra in which $(y, x, x) \equiv (y, x, x^2) \equiv 0 \bmod F$ for all x, y in A, then $(y, x, x) = (y, x, x^2) = 0$ for all x, y in A.*

Proof. From the 5-identity we have $y(x, x, x) + (y, x, x)x = (yx, x, x) - (y, x^2, x) + (y, x, x^2)$ and the right side is in $F \cdot 1$ by hypothesis. Therefore we have

$$y(x, x, x) + (y, x, x)x \equiv 0 \bmod F \cdot 1 \quad \text{for all } x, y \text{ in } A. \tag{5}$$

Now we may assume that there exists a y in A which is linearly independent of 1 and x for otherwise A would be associative. Thus $(x, x, x) = 0$ for all x in A. Consequently $(y, x, x)x \in F \cdot 1 \cap F \cdot x$ for all y in A from which we conclude that $(y, x, x) = 0$.

Irrespective of the number of elements in F we may linearize the identity $(y, x^2, x) \equiv 0 \bmod F$ to get $2(y, x^3, x) \equiv 0 \bmod F$. Therefore the right side of

$$y(x, x^2, x) + (y, x, x^2)x = (yx, x^2, x) - (y, x^3, x) + (y, x, x^3)$$

is in $F \cdot 1$ so that we have $y(x, x^2, x) + (y, x, x^2)x \equiv 0 \bmod F$ for all x, y in A and by the same argument as before we arrive at $(y, x, x^2) = 0$.

We remark that if F has at least three elements and $(y, x, x) \equiv (y, x, x^2) \equiv 0 \bmod F$, then the algebra is strongly right alternative. For by the lemma, $(y, x, x) = (y, x, x^2) = 0$ for all x, y in A, and since F has at least three elements, these identities hold in all extensions.

We are now able to prove our first main result.

Theorem 1. *If A is an algebra in which $(x, x, y) \equiv (y, x, x) \equiv 0 \bmod F$ for all x, y in F and if the characteristic of $F \neq 3$, then A is an alternative algebra.*

Proof. From the 5-identity and Lemma 2 we have $x(y, x, x) \equiv (x, y, x^2)$ mod F and $x(x, x, y) \equiv (x^2, x, y) - (x, x^2, y)$ mod F. We add these congruences to get

$$x[(y, x, x) + (x, x, y)] \equiv 3(x, y, x^2) \bmod F$$

or, again by Lemma 2, $3(x, y, x^2) \equiv 0$ mod F. Since characteristic $F \neq 3$, we have $(x, y, x^2) \equiv 0$ mod F. Then from Lemma 3 it follows that $(y, x, x) = 0$. Linearization of the flexible law gives $(x, x, y) = 0$. Therefore A is alternative.

3. The Jordan case. Recall that a ring R is called noncommutative Jordan if it is flexible and satisfies the Jordan law $(x, y, x^2) = 0$. If R is a ring in which to each $a \in R$ there is a unique $b \in R$ such that $2b = a$, we can define the attached ring R^+ to be the same additive group as R with multiplication in R^+ given by $x \cdot y = 1/2(xy + yx)$ where xy denotes the multiplication in R. In particular this applies to an algebra over a field of characteristic $\neq 2$.

Theorem 2. *If A is an algebra over a field F of characteristic $\neq 2$ in which $[x, y] \equiv (x, y, x^2) \equiv 0$ mod F for all x, y in A, then A is a Jordan algebra.*

Proof. Since A contains an identity element 1, linearization of $(x, y, x^2) \equiv 0$ mod F yields $(x, y, x) \equiv 0$ mod F [8]. On the other hand $[xy, x] = x[y, x] + (x, y, x)$ holds in any ring. Therefore we get $x[y, x] \in F \cdot 1 \cap F \cdot x$ so that we can conclude that $[x, y] = 0$ and A is commutative. Hence A is flexible and

$$(x, y, z) + (y, z, x) + (z, x, y) = 0 \quad \text{for all } x, y, z \text{ in } A. \tag{6}$$

Thus $(x^2, y^2, x) = -(y^2, x, x^2) - (x, x^2, y^2)$. Linearization of the hypothesis gives $(x^2, y, z) + 2(xz, y, x) \equiv 0$ mod F and thus the right side of the last equation is congruent to $2(x^2y, x, y) + 2(y, x^2, xy)$ mod F. Thus we have

$$(x^2, y^2, x) \equiv 2(x^2y, x, y) + 2(y, x^2, xy) \bmod F \quad \text{for all } x, y \text{ in } A. \tag{7}$$

Now by (6) again we have

$$2(x^2y, x, y) + 2(y, x^2, xy) = 2(x^2y, x, y) - 2(x^2, yx, y) + 2(x^2, y, xy).$$

Therefore, we have

$$(x^2, y^2, x) \equiv 2(x^2y, x, y) - 2(x^2, yx, y) + 2(x^2, y, xy) \bmod F \quad \text{for all } x, y \text{ in } A. \tag{8}$$

Thus, by the 5-identity and flexibility we arrive at: $(x^2, y^2, x) \equiv 2(x^2, y, x)y \bmod F$. Hence $2(x^2, y, x)y \in F \cdot 1 \cap F \cdot y$ for all x, y in A. It follows that $(x^2, y, x) = 0$ and A is a Jordan algebra.

An algebra R is called Jordan admissible if R^+ is a Jordan algebra.

Corollary 1. *If A is an algebra over a field F of characteristic $\neq 2$ in which $(x, y, x^2) \equiv 0 \bmod F$ for all x, y in A, then A is a Jordan admissible algebra.*

Proof. Again it follows from the hypothesis that $(x, y, x) \equiv 0 \bmod F$. Therefore Lemma 1 yields $(x, x, x) = 0$ and its linearized version $(x, x, y) + (x, y, x) + (y, x, x) = 0$. Now it is straightforward that

$$4[(x \cdot y) \cdot x^2 - x \cdot (y \cdot x^2)] = (x, y, x^2) - (x^2, y, x) + (y, x, x^2) - (x^2, x, y) + (x, x^2, y) - (y, x^2, x).$$

Hence

$$\begin{aligned} 4[(x \cdot y) \cdot x^2 - x \cdot (y \cdot x^2)] &\equiv 2(y, x, x^2) - 2(y, x^2, x) \bmod F \quad (\text{since } (x, y, x) \in F \cdot 1) \\ &\equiv -2(yx, x, x) + 2(y, x, x)x \bmod F \quad (5\text{-identity}) \\ &\equiv 2(x, x, yx) + 2(y, x, x)x \bmod F \\ &\equiv 2x(x, y, x) + 2(x, x, y)x + 2(y, x, x)x \bmod F \quad (5\text{-identity}) \\ &\equiv 2[(x, y, x) + (x, x, y) + (y, x, x)]x \bmod F \\ &= 0 \end{aligned}$$

since $(x, y, x) + (x, x, y) + (y, x, x) = 0$. Therefore $(x \cdot y) \cdot x^2 - x \cdot (y \cdot x^2) \equiv 0 \bmod F$. Hence A^+ satisfies the conditions of Theorem 2 and is a Jordan algebra. Therefore A is Jordan admissible.

Since a flexible, Jordan admissible algebra is noncommutative Jordan [7], the following corollary is immediate.

Corollary 2. *If A is a flexible algebra over a field F of characteristic $\neq 2$ in which $(x, y, x^2) \equiv 0 \bmod F$ for all x, y in A, then A is a noncommutative Jordan algebra.*

Examples. 1. The following example shows that the result of Corollary

2 is not true if the algebra is not assumed to be flexible. Let A be the 4-dimensional algebra over a field F of characteristic $\neq 2$ with basis $1, a, b, c$. Define multiplication by $a^2 = b^2 = c^2 = 1$, $ab = -ba = c$, and all other products zero. Then, for all x, y in A one notes that $(x, y, x) \equiv (x^2, y, x) \equiv 0 \mod F$. However $(a, b, c) + (c, b, a) = 2 \neq 0$. Therefore A is not flexible. In addition it is easy to see that A is a simple algebra.

2. There are many examples of simple, power-associative algebras in which $[x, y] \equiv 0 \mod F$ but which are not commutative. See, for example, Example 2 of [4] and the class of algebras constructed in [1].

3. The following is an example of an algebra A with an idempotent e in the center C of A such that $(x, y, z) \in F \cdot e \subseteq C$ for all x, y, z in A, but A is not even power-associative. Let A be the 4-dimensional algebra with basis e, x, y, z and multiplication given by: $xy = yx = z$, $e^2 = zx = xz = yz = zy = e$ and all other products zero. Thus, the results of Theorem 1 would be false if the congruences were assumed modulo the center.

4. **The power-associative case.** In an arbitrary algebra A powers of elements x in A are defined inductively by $x^n = xx^{n-1}$. A is called power-associative if $x^m x^n = x^{m+n}$ for all x in A and for all positive integers m, n. This is easily equivalent to $(x^p, x^q, x^r) = 0$ for all x, p, q, r.

Theorem 3. *If A is an algebra in which $(x^p, x^q, x^r) \equiv 0 \mod F$ for all x in A and all positive integers p, q, r, then A is a power-associative algebra.*

Proof. Let x be in A. Then by the 5-identity we have

$$x(x, x, x^2) + (x, x, x)x^2 = (x^2, x, x^2) - (x, x^2, x^2) + (x, x, x^3).$$

By hypothesis the right side is in $F \cdot 1$. Thus $x(x, x, x^2) + (x, x, x)x^2 \equiv 0 \mod F$. It follows that either $(x, x, x) = 0$ or $x^2 \in F \cdot 1 + F \cdot x$. But the latter also implies that $(x, x, x) = 0$. Assume inductively that $x^m x^n = x^{m+n}$ for $3 \leq m + n \leq N$. Now let $m + n = N + 1$. If $m = 1$, then $x^m x^n = x^{m+n}$ by definition. If $m > 1$, then by the induction hypothesis we have

$$x^m x^n = (xx^{m-1})x^n = (x, x^{m-1}, x^n) + x^{N+1}. \tag{9}$$

Thus, if $(x, x^{m-1}, x^n) = 0$ we are done. By the 5-identity we have $x(x^{m-1}, x^n, x^2) + (x, x^{m-1}, x^n)x^2 \equiv 0 \mod F$. Now if $(x, x^{m-1}, x^n) \neq 0$, then $x^2 \in F \cdot 1 + F \cdot x$ which implies that $x^n \in F \cdot 1 + F \cdot x$ and $x^{m-1} \in F \cdot 1 + F \cdot x$. Thus, $(x, x^{m-1}, x^n) = 0$ (since $(x, x, x) = 0$). Therefore $x^m x^n = x^{N+1} = x^{m+n}$ and A is power-associative.

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