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INTERSECTING UNIONS OF CONVEX SETS IN R^n

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ABSTRACT. Let $\mathcal{C} = \{C_\alpha: \alpha$ in some index set $I\}$ be a collection of convex sets, and let $\mathfrak{M} = \{C_\alpha \cup C_\beta: \alpha \neq \beta, C_\alpha, C_\beta$ in $\mathcal{C}\}$. In this paper, various decomposition theorems are obtained for the set $\bigcap \mathfrak{M}$.

1. **Introduction.** In [1], it is proved that if $\mathcal{C}$ is a collection of closed convex sets in the plane and if $\mathfrak{M} = \{A \cup B: A, B$ distinct members of $\mathcal{C}\}$, then the set $\bigcap \mathfrak{M}$ is expressible as a union of three or fewer closed convex sets. In this paper, an attempt is made to obtain similar decompositions without the restriction that $\mathcal{C}$ be planar. Although several theorems are stated for an arbitrary linear topological space, restrictions on the convex sets reduce the setting to R^n, and all the theorems are essentially finite dimensional ones. Throughout the paper, aff S and ker S will be used to denote the affine hull and kernel, respectively, for the set S. If S is convex, dim S will denote the dimension of the affine hull of S, and for convenience, we say that the dimension of the null set is -1.

2. **Decomposition theorems for $\bigcap \mathfrak{M}$.** The following easy lemmas will be useful.

Lemma 1. *Let $\mathcal{C} = \{C_\alpha: \alpha$ in some index set $I\}$ be a collection of sets, and let $\mathfrak{M} = \{C_{\alpha_1} \cup \dots \cup C_{\alpha_k}: \alpha_1, \dots, \alpha_k$ distinct member of $I\}$. Then $x \in \bigcap \mathfrak{M}$ if and only if there are at most $k-1$ members α in I for which $x \notin C_\alpha$.*

Lemma 2. *Let $\mathcal{C} = \{C_\alpha: \alpha$ in some index set $I\}$ be a collection of convex sets in some linear topological space, and let $\mathfrak{M} = \{C_{\alpha_1} \cup \dots \cup C_{\alpha_k}: \alpha_1, \dots, \alpha_k$ distinct members of $I\}$. Then $\bigcap \mathcal{C} \subseteq \ker(\bigcap \mathfrak{M})$.*

Theorem 1. *Let $\mathcal{C} = \{C_\alpha: \alpha$ in some index set $I\}$ be a collection of convex sets in some linear topological space, and assume that, for some $n \geq 1$, at least $n+1$ of these sets have dimension no greater than $n-1$.*

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Letting $\mathfrak{M} = \{C_\alpha \cup C_\beta : \alpha \neq \beta,\ C_\alpha,\ C_\beta$ *in* $\mathcal{C}\}$, *if* $\dim \operatorname{aff}(\bigcap \mathfrak{M})$ *is at least* n, *then* $\bigcap \mathfrak{M}$ *is a union of* $n+1$ *or fewer convex sets, each containing* $\bigcap \mathcal{C}$. *The number* $n+1$ *is best possible for every* n.

Proof. We use an inductive argument. If $n = 1$, then at least two members A, B of $\mathcal{C}$ are singleton sets, $\bigcap \mathfrak{M} \subseteq A \cup B$, and trivially $\bigcap \mathfrak{M}$ consists of exactly two points.

Assume that the result is true for every integer m, $1 \leq m \leq n-1$, to prove for n. There are two cases to consider.

Case 1. Suppose that there are $n+1$ affinely independent points $x_1, \ldots, x_{n+1}$ of $\bigcap \mathfrak{M}$ not in $\bigcap \mathcal{C}$. Then for each i, $1 \leq i \leq n+1$, we may select a corresponding set A_i in $\mathcal{C}$ with $x_i \notin A_i$. For any C in $\mathcal{C} \sim \{A_1, \ldots, A_{n+1}\}$, C necessarily contains each of the $n+1$ affinely independent points $x_1, \ldots, x_{n+1}$, and so $\dim C \geq n$. Hence the A_i sets must be exactly those members of $\mathcal{C}$ which have dimension no greater than $n-1$, and the A_i sets are necessarily distinct, $1 \leq i \leq n+1$. Then each A_i must contain each of the n points x_j, $j \neq i$, $1 \leq j \leq n+1$. Since the points $x_1, \ldots, x_{n+1}$ are vertices of an n-dimensional simplex, each A_i lies in the affine hull of a facet of the simplex. Therefore $A_1 \cap \cdots \cap A_{n+1} = \emptyset$ and $\bigcap \mathfrak{M}$ is just the union of the $n+1$ convex sets B_i, where $B_i \equiv \bigcap \{C : C \text{ in } \mathcal{C},\ C \neq A_i\} = \{x_i\}$, $1 \leq i \leq n+1$.

Case 2. If there are at most $k+1 < n+1$ affinely independent points $x_1, \ldots, x_{k+1}$ of $\bigcap \mathfrak{M}$ not in $\bigcap \mathcal{C}$, these points lie in a k-dimensional flat π (and clearly we may assume $0 \leq k$ for otherwise the result is trivial). Select points $x_{k+2}, \ldots, x_{n+1}$ in $\bigcap \mathfrak{M}$ so that $x_1, \ldots, x_{k+1}, x_{k+2}, \ldots, x_{n+1}$ are affinely independent. Then each of the $n-k$ points $x_{k+2}, \ldots, x_{n+1}$ must lie in $\bigcap \mathcal{C}$. For each of the members A of $\mathcal{C}$ for which $\dim A \leq n-1$, there are no more than $n-(n-k) = k$ affinely independent points of A in π, and $\dim(A \cap \pi) \leq k-1$. Hence $\mathcal{C}' \equiv \{C \cap \pi : C \text{ in } \mathcal{C}\}$ is a collection of convex sets, at least $n+1 > k+1$ of which have dimension no greater than $k-1$. Letting $\mathfrak{M}' \equiv \{C'_\alpha \cup C'_\beta : \alpha \neq \beta,\ C'_\alpha,\ C'_\beta \text{ in } \mathcal{C}'\}$, $\dim \operatorname{aff}(\bigcap \mathfrak{M}') = \dim \operatorname{aff}((\bigcap \mathfrak{M}) \cap \pi) = k$. Therefore, by our induction hypothesis, $\bigcap \mathfrak{M}'$ is a union of $k+1$ or fewer convex sets, say $S'_1, \ldots, S'_{k+1}$, each containing $\bigcap \mathcal{C}'$.

We assert that $\bigcap \mathfrak{M}$ is a union of the $k+1$ convex sets $S_i \equiv S'_i \cup (\bigcap \mathcal{C})$, $1 \leq i \leq k+1$: For x in $\bigcap \mathfrak{M}$ and x not in any S'_i, $1 \leq i \leq k+1$, then $x \notin \pi$, so x must belong to every C in $\mathcal{C}$. Hence $S_1 \cup \cdots \cup S_{k+1} = \bigcap \mathfrak{M}$. To show that each S_i is convex, clearly we need only consider r in S'_i, s in $\bigcap \mathcal{C}$

to show that $[s, r] \subseteq S_i$. Now by Lemma 2, s is in $\ker(\bigcap \mathfrak{M})$, so $[s, r] \subseteq \bigcap \mathfrak{M}$. If $s \in \pi$, the result is immediate since $\bigcap \mathcal{C}' \subseteq S_i$. Otherwise, $[s, r) \cap \pi = \emptyset$, so $[s, r) \subseteq \bigcap \mathfrak{M} \sim \pi \subseteq \bigcap \mathcal{C}$, and $[s, r] \subseteq (\bigcap \mathcal{C}) \cup S_i' = S_i$. Thus S_i is convex, $1 \le i \le k + 1$, and the assertion is proved, finishing Case 2.

This completes the inductive argument, and we conclude that the statement of the theorem is true for every integer $n \geq 1$.

Remark. To see that the bound of $n + 1$ in Theorem 1 is best possible, refer to Example 1 of this paper.

Theorem 2. *Let $\mathcal{C} = \{C_\alpha : \alpha$ in some index set $I\}$ be a collection of convex sets in R^n, $n \geq 1$, and let $\mathfrak{M} = \{C_\alpha \cup C_\beta : \alpha \neq \beta, C_\alpha, C_\beta$ in $\mathcal{C}\}$. If there is an $n + 1$ member subset J of I such that $\operatorname{aff}(C_\alpha \cap (\bigcap \mathfrak{M})) \neq \operatorname{aff}(C_\beta \cap (\bigcap \mathfrak{M}))$ for $\alpha \neq \beta$, α in J, β in I, then $\bigcap \mathfrak{M}$ is a union of $n + 1$ or fewer convex sets, each containing $\bigcap \mathcal{C}$. The number $n + 1$ is best possible.*

Proof. The inductive argument of Theorem 1 may be suitably adapted to yield the result. The only significant difference appears in Case 2: As in Case 2, affinely independent points $x_1, \ldots, x_{k+1}, x_{k+2}, \ldots, x_{j+1}$ are selected in $\bigcap \mathfrak{M}$ with $x_1, \ldots, x_{k+1}$ in the k-dimensional flat π and not in $\bigcap \mathcal{C}$, and $x_{k+2}, \ldots, x_{j+1}$ in $\bigcap \mathcal{C} \sim \pi$, where $j = \dim \operatorname{aff}(\bigcap \mathfrak{M})$ and $0 \le k \le j$. Then for α in J, β in I, and $\alpha \neq \beta$,

$$\operatorname{aff}\left(C_\alpha \cap \pi \cap \left(\bigcap \mathfrak{M}\right)\right) \neq \operatorname{aff}\left(C_\beta \cap \pi \cap \left(\bigcap \mathfrak{M}\right)\right),$$

for otherwise

$$\operatorname{aff}\left(\left[C_\alpha \cap \pi \cap \left(\bigcap \mathfrak{M}\right)\right] \cup \{x_{k+2}, \ldots, x_{j+1}\}\right)$$
$$= \operatorname{aff}\left(\left[C_\beta \cap \pi \cap \left(\bigcap \mathfrak{M}\right)\right] \cup \{x_{k+2}, \ldots, x_{j+1}\}\right),$$

and since $x_{k+2}, \ldots, x_{j+1}$ are in every C in $\mathcal{C}$,

$$\operatorname{aff}\left(C_\alpha \cap \left(\bigcap \mathfrak{M}\right)\right) = \operatorname{aff}\left(C_\beta \cap \left(\bigcap \mathfrak{M}\right)\right),$$

clearly impossible. Hence the induction hypothesis may be applied to the sets $\mathcal{C}'$ and $\bigcap \mathfrak{M}'$ of Case 2 to complete the argument.

The following example shows that the bound of $n + 1$ in Theorems 1 and 2 is best possible.

Example 1. For $n \geq 1$, let T denote an n-dimensional simplex and $\mathcal{C}$ the collection of facets of T. Then $\mathcal{C}$ has $n + 1$ members, $\bigcap \mathcal{C} = \emptyset$, and $\bigcap \mathfrak{M}$ is the collection of points which lie in exactly n facets of T. Hence

$\bigcap\mathfrak{M}$ is just the vertex set of T and consists of $n+1$ isolated points.

Another kind of decomposition is given in Theorem 3.

Theorem 3. *Let $\mathcal{C} = \{C_\alpha: \alpha$ in some index set $I\}$ be a collection of closed convex sets, and let $\mathfrak{M} = \{C_\alpha \cup C_\beta: \alpha \neq \beta, C_\alpha, C_\beta$ in $\mathcal{C}\}$. If for some $k \geq 1$ members $\alpha_1, \dots, \alpha_k$ in I, $\dim(C_{\alpha_1} \cap \dots \cap C_{\alpha_k}) \leq i$, $-1 \leq i \leq 2$, then $\bigcap\mathfrak{M}$ is a union of $k+i+1$ or fewer closed convex sets. The bound is best possible for every pair k, i.*

Proof. For convenience of notation, let $C_{\alpha_i} = C_i$, $1 \leq i \leq k$, and define $D_i \equiv \bigcap\{C: C$ in $\mathcal{C}$, $C \neq C_i\}$. For x in $\bigcap\mathfrak{M}$, either x lies in one of the closed convex sets D_i, $1 \leq i \leq k$, or $x \in C_1 \cap \dots \cap C_k$.

We assert that the set $C_1 \cap \dots \cap C_k \cap (\bigcap\mathfrak{M})$ is expressible as a union of $i+1$ or fewer closed convex sets: Define $\mathcal{C}' \equiv \{C_1 \cap \dots \cap C_k \cap C_\alpha = C'_\alpha: \alpha$ in $I\}$, and let $\mathfrak{M}' \equiv \{C'_\alpha \cup C'_\beta: \alpha \neq \beta, C'_\alpha, C'_\beta$ in $\mathcal{C}'\}$. Then $C_1 \cap \dots \cap C_k \cap (\bigcap\mathfrak{M})$ is exactly $\bigcap\mathfrak{M}'$. If $i = 2$, then $\mathcal{C}'$ is a collection of closed convex sets in the plane, and by suitably adapting Theorem 1 in [1], $\bigcap\mathfrak{M}'$ is a union of three or fewer closed convex sets, the desired result. In case $i = 1$, techniques used in [1] may be used to show that $\bigcap\mathfrak{M}'$ is a union of 2 or fewer closed convex sets. For $i = 0$ or $i = -1$, the result is trivial.

Therefore, $C_1 \cap \dots \cap C_k \cap (\bigcap\mathfrak{M})$ is a union of $i+1$ closed convex sets, and hence $\bigcap\mathfrak{M}$ is a union of $k+i+1$ or fewer closed convex sets, finishing the proof of Theorem 3.

Example 2 reveals that the bound $k+i+1$ is best possible for every pair k, i.

Example 2. For a given $k \geq 1$ and for $-1 \leq i \leq 2$, if $k+i \geq 1$, let $\mathcal{C}$ denote the $k+i+1$ facets of a simplex T in R^{k+i}. Then k members of $\mathcal{C}$ intersect in an i-dimensional set, and $\bigcap\mathfrak{M}$, the vertex set of T, is a union of $k+i+1$ closed convex sets. If $k+i=0$, some member of $\mathcal{C}$ is empty, and $\bigcap\mathfrak{M}$ is convex.

Corollary. *If $\mathcal{C}$ is a finite collection of closed convex sets in R^n and $\dim(\bigcap\mathcal{C}) \leq 2$, then the corresponding set $\bigcap\mathfrak{M}$ is a union of $\sigma(n)+3$ or fewer closed convex sets, where $\sigma(n) = \max(n+1, 2n-4)$.*

Proof. By a theorem of Katchalski [2], if all $\sigma(n)$ sets in $\mathcal{C}$ have at least a 3-dimensional intersection, then so does $\bigcap\mathcal{C}$. Hence if $\dim(\bigcap\mathcal{C}) \leq 2$, there are some $\sigma(n)$ sets in $\mathcal{C}$ whose intersection has dimension no more than 2. By Theorem 3, $\bigcap\mathfrak{M}$ is a union of $\sigma(n)+2+1$ or fewer closed convex sets.

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A CHARACTERIZATION OF THE KERNEL OF A CLOSED SET

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ABSTRACT. Let S be a closed subset of some linear topological space such that int ker $S \neq \emptyset$ and ker $S \neq S$. Let $\mathcal{C}$ denote the collection of all maximal convex subsets of S and, for any fixed $k \geq 1$, let $\mathfrak{M} = \{A_1 \cup \cdots \cup A_k : A_1, \ldots, A_k$ distinct members of $\mathcal{C}\}$. Then $\mathfrak{M} \neq \emptyset$ and $\bigcap \mathfrak{M} = \ker S$.

If $\mathcal{C}$ is the collection of all maximal convex subsets of some set S, it is easy to show that $\bigcap \mathcal{C} = \ker S$. This paper provides an interesting and perhaps surprising analogue of this well-known result. Throughout the paper, conv S, int S, and ker S will be used to denote the convex hull, interior, and kernel, respectively, for the set S.

Further, we will make use of these familiar definitions: For points x, y in a set S, we say *x sees y via S* if and only if the corresponding segment $[x, y]$ lies in S. A subset T of S is said to be a *visually independent subset of S* if and only if for every x, y in T, $x \neq y$, x does not see y via S.

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