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T-REGULAR-CLOSED CONVERGENCE SPACES

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ABSTRACT. It is known that a convergence space which has a regular compactification is almost identical to a completely regular topological space. It is shown that a less restricitive class of convergence spaces have T-regular-closed extensions with the universal property of the Stone-Čech compactification.

1. Introduction. In [6] one of us showed that each Hausdorff convergence space has a Hausdorff compactification with an extension property for continuous functions into compact regular spaces. In [7] two of us showed that a convergence space has a regular compactification iff the space is regular and has the same ultrafilter convergence as a completely regular topological space. In this paper we obtain a "regular compactification" which has a universal property like the compactifications of [6] and [7] for a class of convergence spaces (which includes spaces having highly nonidempotent closure operators) by relinquishing the requirement that the "compactification" space be compact. Instead, we require that it be T-regular-closed, a concept resembling, but more general than, compactness.

A *convergence space* $(X, \to)$ consists of a set X and a relation "$\to$" between the filters on X and the elements of X, subject to the following conditions:

(1) $\dot{x} \to x$, all $x \in X$;

(2) $\mathcal{F} \to x$ and $\mathcal{F} \leq \mathcal{G}$ implies $\mathcal{G} \to x$;

(3) $\mathcal{F} \to x$ implies $\mathcal{F} \cap \dot{x} \to x$.

For $x \in X$, $\dot{x}$ denotes the fixed ultrafilter generated by $\{x\}$; if A is a nonempty subset of X, then $\dot{A}$ will denote the filter of all oversets of A. The expression "$\mathcal{F} \to x$" should be read "the filter $\mathcal{F}$ converges to the point x". We will use the abbreviation "u.f." for "ultrafilter".

Throughout the paper, *space* will mean *convergence space*. We will

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usually refer to a space as "X" rather than "$(X, \to)$". A space is *Hausdorff* if each filter converges to at most one point. *All spaces are assumed to be Hausdorff unless otherwise indicated.*

Fischer [3] defined a space X to be *regular* if $\mathrm{cl}_X\, \mathcal{F} \to x$ whenever $\mathcal{F} \to x$, where "cl_X" is the closure operator for X and $\mathcal{F}$ is a filter on X. This definition has become standard. We define X to be *T-regular* if $\mathcal{F} \to x$ implies $\mathrm{cl}_{\lambda X}\, \mathcal{F} \to x$; λX, the *topological modification* of X, is the space consisting of the set X equipped with the finest topology coarser than the convergence structure of X. A subset A is λX-closed iff $A = \mathrm{cl}_X A$. A T-regular space is clearly regular, and the two versions of regularity are equivalent if X is a topological space. Some examples are given in the next section of important classes of T-regular spaces. Like regularity, T-regularity is productive and hereditary.

A T-regular space will be called *T-regular-closed* if X is a closed subset of any T-regular space in which it can be embedded. A compact T-regular space is obviously T-regular-closed; [4, Example 3.10] describes a T-regular-closed space which is not compact. A study of T-regular-closed spaces is given in §3; we show, among other things, that a continuous function from a T-regular-closed space onto a T-regular space is closed. §4 is concerned with embedding T-regular spaces in T-regular-closed spaces.

2. **Examples.** The following classes of spaces are shown to be T-regular: c-embedded spaces, locally compact regular spaces, and lattices with order convergence.

Let X be a space, $C_c(X)$ the set of continuous real-valued functions on X with the coarsest convergence structure (called *continuous convergence*) relative to which the natural map ω: $C_c(X) \times X \to R$, defined by $\omega(f, x) = f(x)$, is continuous. (R denotes the real line with its usual topology.) X is *c-embedded* if the evaluation map i: $X \to C_c C_c(X)$, defined by $i(x)(f) = f(x)$, all $f \in C(X)$, is an embedding. Feldman [2] has proved that all c-embedded spaces are T-regular.

A convergence space X is said to be *locally compact* if each convergent filter contains a compact set.

Proposition 2.1. *A locally compact regular space X is T-regular.*

Proof. Let $\mathcal{F} \to x$. Then $\mathcal{F}$ contains a set A which is compact and hence closed. As a subspace of X, A is a compact regular space, and it follows from [7, Lemma 1] that cl_X and $\mathrm{cl}_{\lambda X}$ coincide for subsets of A; thus $\mathrm{cl}_X\, \mathcal{F} = \mathrm{cl}_{\lambda X}\, \mathcal{F}$. Since X is regular, $\mathrm{cl}_{\lambda X}\, \mathcal{F} \to x$.

Pervin and Biesterfeldt [5] have shown that a lattice with order convergence is regular. We will give a shorter proof showing that such a space is T-regular. We use the filter characterizations of order convergence given by Ward [9]. Let X be a partially ordered set, $\mathfrak{F}$ a filter on X. Let $L(\mathfrak{F}) = \{x \in X$, there exists $F \in \mathfrak{F}$ such that $x \le y$ for all $y \in F\}$, and let $U(\mathfrak{F})$ be defined dually. $\mathfrak{F}$ *order converges* to x if $x = \inf U(\mathfrak{F}) = \sup L(\mathfrak{F})$. It is well known that order convergence need not be topological, even in a complete lattice.

Proposition 2.2. *Order convergence in any lattice X is T-regular.*

Proof. Let $\mathfrak{F}$ order converge to x. Let $\mathfrak{G}$ be the filter generated by all sets of the form $[a, b] = \{y: a \le y \le b\}$, for $a \in L(\mathfrak{F})$ and $b \in U(\mathfrak{F})$. Since $L(\mathfrak{G}) = L(\mathfrak{F})$ and $U(\mathfrak{G}) = U(\mathfrak{F})$, it follows immediately that $\mathfrak{G}$ order converges to x. Also, sets of the form $[a, b]$ are closed in the interval topology on X (see [9]), which is known to be coarser than order convergence. Thus $\mathrm{cl}_{\lambda X} \mathfrak{F} \ge \mathfrak{G}$, and so $\mathrm{cl}_{\lambda X} \mathfrak{F} \to x$.

3. **T-regular-closed spaces.** Regular-closed topological spaces have been investigated by a number of mathematicians; for a summary of results on this topic see [1]. Regular-closed convergence spaces (but not T-regular-closed spaces) are studied in [4]. Note that the concept of a regular-closed topological space is not equivalent to that of a topological regular-closed (or T-regular-closed) convergence space, and the results obtained for convergence spaces differ in various ways from those for topological spaces.

The proof of Theorem 3.1 is almost identical to that of Theorem 2.10 of [4] and will therefore be omitted.

Theorem 3.1. *A T-regular space X is T-regular-closed iff, for each filter $\mathfrak{F}$ on X, $\mathrm{cl}_{\lambda X} \mathfrak{F}$ has an adherent point.*

Equivalently, a T-regular space X is T-regular-closed iff each maximal closed filter on X converges.

Corollary 3.2. *A closed subspace of a T-regular-closed space is T-regular-closed.*

Theorem 3.3. *Let f be a continuous function from a T-regular-closed space X onto a T-regular space Y. Then f is a closed map.*

Proof. Let A be a closed subset of X, $y \in \mathrm{cl}_Y fA$. Then there is an u.f. $\mathfrak{F}$ on fA such that $\mathfrak{F} \to y$ in Y. Let $\mathfrak{G}$ be an u.f. on X which is finer than $f^{-1}(\mathfrak{F}) \vee A^{\bullet}$, where $A^{\bullet}$ is the filter of all oversets of A. Since X is

T-regular-closed, there is an adherent point x of $\mathrm{cl}_{\lambda X}\,\mathcal{G}$. However, since A is closed, $A \in \mathrm{cl}_{\lambda X}\,\mathcal{G}$, and so $x \in A$. But $f(x)$ is an adherent point of $\mathrm{cl}_{\lambda Y}\,\mathcal{F}$, and so $f(x) = y$, since $\mathrm{cl}_{\lambda Y}\,\mathcal{F} \to y$. Thus $y \in fA$.

We next consider products of T-regular-closed spaces. The example that follows shows that the property of being T-regular-closed is not productive.

Example 3.4. Let X be a countable infinite set, x a fixed point in X. Since there are 2^c free u.f.'s on each infinite subset of X (c the cardinality of the real line) and only c subsets of X, we can assign to each infinite subset A of X two distinct free u.f.'s, $\mathcal{F}_A$ and $\mathcal{G}_A$ which contain A such that $\mathcal{F}_A \neq \mathcal{F}_B$ if $B \neq A$ and $\mathcal{F}_A \neq \mathcal{G}_B$ for all infinite subsets B of X. Let X_1 be the set X with the finest convergence structure such that $\mathcal{F}_A \to x$ in X_1 for each infinite subset A. Let X_2 be the set X with the finest convergence structure such that $\mathcal{G}_A \to x$ in X_2 for each infinite subset A.

The spaces X_1 and X_2 are clearly T-regular-closed by Theorem 3.1. However, no free u.f. which contains the diagonal in the product space $X_1 \times X_2$ can converge, and so $X_1 \times X_2$ is not T-regular-closed.

Theorem 3.5. *If X_1 and X_2 are T-regular-closed, then $X_1 \times X_2$ is T-regular-closed iff both projection maps are closed.*

Proof. The condition is necessary by Theorem 3.3. Conversely, let $\mathcal{F}$ be a filter on $Y = X_1 \times X_2$; we must show that $\mathrm{cl}_{\lambda Y}\,\mathcal{F}$ has an adherent point. Since the first projection map P_1 is a closed map, $P_1 \mathrm{cl}_{\lambda Y}\,\mathcal{F} = \mathrm{cl}_{\lambda X_1}\,P_1\mathcal{F}$, and by hypothesis there is a filter $\mathcal{G}$ finer than $P_1\mathrm{cl}_{\lambda Y}\,\mathcal{F}$ such that $\mathcal{G} \to x$, for some $x \in X_1$. Hence, $\mathcal{H} = (P_1^{-1}\mathrm{cl}_{\lambda X_1}\,\mathcal{G}) \vee \mathrm{cl}_{\lambda Y}\,\mathcal{F}$ is a filter on Y, and $\mathrm{cl}_{\lambda Y}\,\mathcal{H} = \mathcal{H}$. Since the second projection map P_2 is closed, $\mathrm{cl}_{\lambda X_2}\,P_2\mathcal{H} = P_2\mathcal{H}$. Let $\mathcal{K}$ be an u.f. finer than $P_2\mathcal{H}$ such that $\mathcal{K} \to y$, for some $y \in X_2$. Then $\mathcal{H} \vee P_2^{-1}\mathcal{K} = \mathcal{H}_1$ is a filter on Y. Also, $P_1\mathcal{H}_1 \geq \mathrm{cl}_{\lambda X_1}\,\mathcal{G}$, and so $P_1\mathcal{H} \to x$ in X_1. Similarly, $P_2\mathcal{H}_1 \geq \mathcal{K}$, and so $P_2\mathcal{H}_1 \to y$ in X_2. Hence, $\mathcal{H}_1 \to (x, y)$ in Y, and so (x, y) is adherent to $\mathrm{cl}_{\lambda Y}\,\mathcal{F}$.

Lemma 3.6. *If X_1 is compact regular and X_2 is T-regular-closed, then the product is T-regular-closed.*

Proof. It is easy to see that the second projection map P_2 is closed. Let $\mathcal{F}$ be a filter on $Y = X_1 \times X_2$. By hypothesis, $\mathrm{cl}_{\lambda X_2}\,P_2\mathcal{F}$ has an adherent point y. Hence there is a filter $\mathcal{G}$ finer that $\mathrm{cl}_{\lambda X_2}\,P_2\mathcal{F} = P_2\mathrm{cl}_{\lambda Y}\,\mathcal{F}$, and $\mathcal{G}$ converges in X_2 to y. Let $\mathcal{H}$ be an u.f. on Y finer than $P_2^{-1}(\mathcal{G}) \vee \mathrm{cl}_{\lambda Y}\,\mathcal{F}$. Then $P_1\mathcal{H} \to x$, for some $x \in X_1$, and $P_2\mathcal{H} \to y$ in X_2, so that (x, y) is an adherent point of $\mathrm{cl}_{\lambda Y}\,\mathcal{F}$.

Theorem 3.7. *If both* X_1 *and* X_2 *are locally compact and* T*-regular-closed, then* $X_1 \times X_2$ *is* T*-regular-closed.*

Proof. It is sufficient, by the preceding theorem, to prove that the projection maps P_1 and P_2 are closed. To show that P_1 is closed, let B be closed in $Y = X_1 \times X_2$, and let $x \in \mathrm{cl}_{X_1} P_1 B$. Then there is an u.f. $\mathcal{F}$ on $P_1 B$ which converges to x in X_1 and contains a compact set A. By Lemma 3.6, $A \times X_2$ is T-regular-closed. Also, $B_1 = B \cap (A \times X_2)$ is a closed set. Hence, $\mathcal{G} = (P_1^{-1}\mathrm{cl}_{\lambda X_1} \mathcal{F}) \vee \dot{B_1}$ is such that $\mathrm{cl}_{\lambda Y} \mathcal{G} = \mathcal{G}$, and since $A \times X_2$ belongs to $\mathcal{G}$, $\mathcal{G}$ has an adherent point (x, b). Hence $x \in P_1 \mathrm{cl}_Y B_1 \subset P_1 \mathrm{cl}_Y B$, and so P_1 is a closed map.

For regular-closed topological spaces, Corollary 3.2 and Theorem 3.3 are known to be false, and Lemma 3.6 and Theorem 3.7 are known to be true. The question of whether a product of regular-closed topological spaces is regular-closed is an unsolved problem. We do not know whether or not Theorem 3.5 is valid for regular-closed topological spaces.

4. Embedding theorems.

Theorem 4.1. *Each* T*-regular space* X *can be embedded in a* T*-regular-closed space* X_1.

Proof. Assume that X is not T-regular-closed. Let y be a point not in X, and let $X_1 = X \cup \{y\}$; let X_1 be the set $X \cup \{y\}$, equipped with the finest convergence structure satisfying the following conditions: $\mathcal{F} \to x$ in X_1, for $x \neq y$, iff $\mathcal{F}$ contains X and the restriction of $\mathcal{F}$ to X converges to x in X; $\mathcal{F} \to y$ iff $\mathcal{F} \geq \dot{y} \wedge \mathcal{G}$, where $\mathcal{G}$ is a filter containing X such that $\mathrm{cl}_{\lambda X} \mathcal{G}$ has no adherent point in X. It is easy to verify that X_1 is T-regular, and that X is a subspace of X_1. Let $\mathcal{F}$ be a filter on X_1. If $\mathcal{F} = \dot{y}$, then y is adherent to y. Otherwise, $\mathcal{F}$ has a restriction $\mathcal{F}_1$ to X. If $\mathrm{cl}_{\lambda X} \mathcal{F}_1$ does not have an adherent point in X, then by the construction of X_1, y is adherent to $\mathrm{cl}_{\lambda X_1} \mathcal{F}$. Thus, by Theorem 3.1, X_1 is T-regular-closed. Finally, the assumption that X is not T-regular-closed guarantees that X is dense in X_1.

We shall now consider the problem of finding a class of T-regular spaces in which each member has a T-regular-closed extension with universal property. Let X be a T-regular space such that λX is a completely regular (including Hausdorff) topological space. Let $Y = \beta\lambda X$ denote the Stone-Čech compactification of λX; let ϕ be the embedding map from λX into Y. Define X^* to be the set Y, equipped with the finest convergence structure satisfying the following conditions: (1) If $\mathcal{F}$ is a filter on X^* containing ϕX, then $\mathrm{cl}_Y \mathcal{F} \to x$ in X^* whenever $\phi^{-1}\mathcal{F} \to \phi^{-1}x$ in X; (2) If $\mathcal{G}$ is an u.f. on X^*

which contains ϕX and $\phi^{-1}\mathcal{G}$ fails to λX-converge, then $\mathrm{cl}_Y\, \mathcal{G}$ converges in X^* to the same point to which it converges in Y.

Theorem 4.2. *If X is a T-regular space and λX is completely regular, then X^* is T-regular and the function $\phi: X \to X^*$ is a dense embedding. Furthermore, if $f: X \to Z$ is a continuous map into a compact regular space Z, then f has a unique continuous extension $f^*: X^* \to Z$.*

Proof. First note that X^* is finer than Y; from this fact and the construction of X^* it is clear that X^* is T-regular. The function ϕ is certainly an injection. The first condition in the definition of X^* guarantees that $\phi: X \to X^*$ is continuous; the second guarantees that ϕX is dense in X^*. If $\mathcal{G}$ is a filter on X^* which contains ϕX and $\mathcal{G} \to x$ in X^*, then, by Condition 1, $\mathcal{G} \geq \mathrm{cl}_Y\, \mathcal{F}$, where $\phi X \in \mathcal{F}$ and $\phi^{-1}\mathcal{F} \to \phi^{-1}x$ in X. Since X is T-regular, $\mathrm{cl}_{\lambda X}\, \phi^{-1}\mathcal{F} \to \phi^{-1}x$ in X. But $\phi^{-1}\mathcal{G} \geq \phi^{-1}\mathrm{cl}_{\lambda X^*}\, \mathcal{F} = \mathrm{cl}_{\lambda X}\, \phi^{-1}\mathcal{F}$, and so $\phi^{-1}\mathcal{G} \to \phi^{-1}x$ in X. Thus ϕ^{-1} is continuous, and $\phi: X \to X^*$ is a dense embedding.

Finally, consider $f: X \to Z$. Then $f: \lambda X \to \lambda Z$ is continuous. By [7, Proposition 1], λZ is a compact Hausdorff topological space. Thus f has an extension $f^*: Y \to \lambda Z$. Let $\mathcal{G} \to x$ in X^*. Assume first that $\mathcal{G} \geq \mathrm{cl}_Y\, \mathcal{F}$, where $\phi^{-1}\mathcal{F} \to \phi^{-1}x$ in X. Then $f^*(\mathcal{F}) = f(\phi^{-1}\mathcal{F}) \to f(\phi^{-1}x) = f^*x$ in Z. Also, $\mathrm{cl}_Z\, f^*\mathcal{F} \to f^*x$ in Z, since Z is regular. But $f^*: Y \to \lambda Z$ is continuous, and so $f^*(\mathrm{cl}_Y\, \mathcal{F}) \geq \mathrm{cl}_Z\, f^*\mathcal{F}$. It follows immediately that $f^*\mathcal{G} \to f^*x$ in Z. Next, assume that $\mathcal{G} \geq \mathrm{cl}_Y\, \mathcal{F}$, where $\mathcal{F}$ is an u.f. on X^* which contains ϕX such that $\phi^{-1}\mathcal{F}$ fails to λX-converge. Then $\mathcal{F} \to x$ in Y, and so $f^*\mathcal{F} \to f^*x$ in λZ. By [7, Theorem 1], Z and λZ have the same ultrafilter convergence, and so $f^*\mathcal{F} \to f^*x$ in Z. By regularity of Z, $\mathrm{cl}_Z f^*\mathcal{F} \to f^*x$, and $f\mathcal{G} \geq f(\mathrm{cl}_Y\, \mathcal{F}) \geq \mathrm{cl}_Z\, f^*\mathcal{F}$, which implies $f^*\mathcal{G} \to f^*x$. Thus we have established that $f^*: X^* \to Z$ is continuous. The uniqueness of the extension is obvious, and so the proof is complete.

Unfortunately, the conditions imposed on X in Theorem 4.2 are not enough to insure that X^* is T-regular-closed. By adding an additional condition, we obtain the desired T-regular-closed extension.

Theorem 4.3. *Let X be a T-regular space such that λX is completely regular and locally compact. Then (X^*, ϕ) is a T-regular-closed extension of X, and each continuous function from X into a compact regular space Z has a continuous closed extension to X^*.*

Proof. In view of Theorems 3.3 and 4.2, it remains only to show that X^* is T-regular-closed. Since λX is locally compact, λX is an open subspace of $Y = \beta\lambda X$. Thus no u.f. on $Y - \phi X$ can converge in Y to a point in ϕX. If $\mathcal{G}$ is an u.f. on $X^* - \phi X$, then $\mathcal{G}$ converges relative to Y to some point z in $Y - \phi X$. Thus there is an u.f. $\mathcal{H}$ on ϕX such that $\mathcal{H} \to z$ in Y, and $\mathcal{G} \geq \mathrm{cl}_Y H$. From the definition of X^*, Condition 2, $\mathcal{G} \to z$ in X^*. These results lead us to conclude that X^* and Y have the same u.f. convergence relative to u.f.'s which contain $Y - \phi X$. Furthermore, if $\mathcal{G} \to x$ in Y and $\phi X \in \mathcal{G}$, then $\phi^{-1}\mathcal{G} \to x$ in λX, and so $\mathcal{G} \to x$ in λX^*. Combining these facts, we arrive at the conclusion that Y and λX^* have the same u.f. convergence, and so λX^* is compact. Thus, $\lambda X^* = Y$.

To show that X^* is T-regular-closed, it is sufficient to show that every maximal closed filter (relative to X^*) converges. By the preceding paragraph, the maximal closed filters relative to X^* are the same as the maximal closed filters relative to Y, and the latter are fixed, since Y is compact. Thus X^* is T-regular-closed, and the proof is complete.

Theorem 4.4. *If X has a regular compactification and λX is locally compact, then X^* coincides with the regular Stone-Čech compactification of X described in* [7].

Proof. From [7, Theorem 1], X is T-regular and λX is completely regular. In the proof of the preceding theorem, it is shown that X^* and Y have the same convergence for u.f.'s containing $Y - \phi X$. By taking this fact into account and comparing the respective constructions, one is led to the desired conclusion.

In [8, §3], a space X_0 is constructed which is shown to be locally compact, c-embedded, and have a completely regular topological modification. However λX_0 is not locally compact, and it can be shown that X_0^* is not T-regular-closed. On the other hand, a simple alteration of this example leads to a space X satisfying the conditions of Theorem 4.3 with the property that the closure operator for X has n-distinct iterations, where n is an arbitrary natural number. Therefore, in contrast to the stringent conditions required in [7] for the existence of a regular compactification, a space can have a T-regular-closed extension as described in Theorem 4.3 and yet bear little resemblence to any topological space.

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