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DISK-LIKE PRODUCTS OF λ CONNECTED CONTINUA. I

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ABSTRACT. A continuum X is λ connected if each two of its points can be joined by a hereditarily decomposable subcontinuum of X. We prove that continua X and Y are atriodic and hereditarily unicoherent when the topological product $X \times Y$ is disk-like. From this result and a theorem of R. H. Bing's it follows that λ connected continua X and Y are arc-like if and only if $X \times Y$ is disk-like.

We call a nondegenerate metric space that is both compact and connected a *continuum*. Let X and Y be continua and let f be a continuous function of X onto Y. If ϵ is a positive number such that for each point p of Y, the diameter of $f^{-1}(p)$ is less than ϵ, then f is said to be an *ϵ-map* of X onto Y.

A continuum X is *arc-like* if for each $\epsilon > 0$ there is an ϵ-map of X onto an arc. Arc-like continua are sometimes called *snake-like* or *chainable*. This property can be described in terms of simple chains of small open sets that cover a space [1].

A continuum X is *disk-like* if for each $\epsilon > 0$ there is an ϵ-map of X onto a disk (2-cell).

A continuum T is called a *triod* if it contains a subcontinuum Z such that $T - Z$ is the union of three nonempty disjoint open sets. When a continuum does not contain a triod, it is said to be *atriodic*.

A continuum is *decomposable* if it is the union of two proper subcontinua. A continuum is *unicoherent* provided that if it is the union of two subcontinua E and F, then $E \cap F$ is connected. A continuum is called *hereditarily decomposable* (*hereditarily unicoherent*) if all of its subcontinua are decomposable (unicoherent).

According to a theorem of R. H. Bing [1, Theorem 11], every atriodic, hereditarily decomposable, hereditarily unicoherent continuum is arc-like.

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For any two metric spaces (X, ψ) and (Y, ϕ), we shall always assume that the distance between two points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ of the topological product $X \times Y$ is defined by

$$\rho(p_1, p_2) = ((\psi(x_1, x_2))^2 + (\phi(y_1, y_2))^2)^{1/2}.$$

Throughout this paper the closure and the boundary of a given set Z are denoted by $\mathrm{Cl}\, Z$ and $\mathrm{Bd}\, Z$ respectively.

Theorem 1. *Suppose that X and Y are continua and that the topological product $X \times Y$ is disk-like. Then X is atriodic and hereditarily unicoherent.*

Proof. Let ψ and ϕ be distance functions for X and Y, respectively, and let D be a disk in a 2-sphere S^2.

Assume that X contains a triod T. It follows that there exist distinct continua B_1, B_2, B_3, and Z such that $T = \bigcup_{i=1}^3 B_i$ and $Z = B_i \cap B_j$ for each i and j $(1 \le i < j \le 3)$. For $i = 1$, 2, and 3, let p_i be a point of $B_i - \bigcup_{j \ne i} B_j$. Define $\{y_i \mid 1 \le i \le 6\}$ to be a set consisting of six distinct points of Y. Let ϵ be the minimum of $\{\phi(y_i, y_j) \mid 1 \le i < j \le 6\} \cup \{\psi(p_i, B_j \cup B_k) \mid 1 \le i \le 3,\ 1 \le j < k \le 3, \text{ and } j \ne i \ne k\}$. Let f be an ϵ-map of $X \times Y$ onto D.

There exist disjoint disks Q_1, Q_2, and Q_3 in S^2 such that for $i = 1$, 2, and 3, Q_i contains $f(\{p_i\} \times Y)$ and misses $f((B_j \cup B_k) \times Y)$ when $1 \le j < k \le 3$ and $j \ne i \ne k$. By staying close to the continuum $f((B_1 \cup B_2) \times \{y_1\})$ we define an arc-segment A_1 in $S^2 - \bigcup_{i=1}^3 Q_i$ such that each component of $Q_1 \cup Q_2$ contains an endpoint of A_1 and $\mathrm{Cl}\, A_1 \cap (\bigcup_{i=2}^6 f(T \times \{y_i\})) = \emptyset$. Define A_2 to be an arc-segment in $S^2 - \bigcup_{i=1}^3 Q_i$ that stays close to $f((B_2 \cup B_3) \times \{y_2\})$ such that $\mathrm{Cl}\, A_2$ meets Q_2 and Q_3 and misses $\mathrm{Cl}\, A_1 \cup \bigcup_{i=3}^6 f(T \times \{y_i\})$. Let A_3 be an arc-segment in $S^2 - \bigcup_{i=1}^3 Q_i$ near $f((B_1 \cup B_3) \times \{y_3\})$ such that $\mathrm{Cl}\, A_3$ meets Q_1 and Q_3 and misses $\mathrm{Cl}(A_1 \cup A_2) \cup \bigcup_{i=4}^6 f(T \times \{y_i\})$.

Note that $\bigcup_{i=1}^3 A_i \cup Q_i$ has exactly two complementary domains in S^2. Hence there exists a complementary domain U of $\bigcup_{i=1}^3 A_i \cup Q_i$ in S^2 that contains two elements of $\{f(Z \times \{y_i\}) \mid 4 \le i \le 6\}$. Assume without loss of generality that $f(Z \times \{y_4\})$ and $f(Z \times \{y_5\})$ are in U. Since Z is a continuum and $f(T \times \{y_4, y_5\}) \cap (\bigcup_{i=1}^3 A_i) = \emptyset$, and since for each point y of Y and $i = 1$, 2, and 3, $f(B_i \times \{y\}) \cap Q_i \ne \emptyset$, it follows that there exist continua H and K in $f(T \times \{y_4\}) \cap \mathrm{Cl}\, U$ and $f(T \times \{y_5\}) \cap \mathrm{Cl}\, U$, respectively, such that for $i = 1$, 2, and 3, $H \cap \mathrm{Bd}\, Q_i \ne \emptyset \ne K \cap \mathrm{Bd}\, Q_i$. But since H and K are disjoint, this is a contradiction [6, Theorem 76, p. 220]. Hence X is atriodic.

Assume that X is not hereditarily unicoherent. It follows that in X there exist continua E and F and nonempty disjoint closed sets A and B such that $E \cap F = A \cup B$. Define C_1 and C_2 to be open subsets of X such that $A \subset C_1$, $B \subset C_2$, and $\mathrm{Cl}\, C_1 \cap \mathrm{Cl}\, C_2 = \emptyset$. Define δ to be a positive number less than $\psi(C_1, C_2)$, $\psi(E, F - (C_1 \cup C_2))$ and $\psi(F, E - (C_1 \cup C_2))$.

We first prove that $E \cup F$ is X. To accomplish this we suppose that there is a point x of $X - (E \cup F)$. Let R be a proper subcontinuum of Y. Let v_1 and v_2 be distinct points of R and let v_3 be a point of $Y - R$. Define δ' to be a positive number less than δ, $\psi(x, E \cup F)$, $\phi(v_1, v_2)$, and $\phi(v_3, R)$. Let g be a δ'-map of $X \times Y$ onto D.

Note that the continua $g(X \times \{v_i\})$ and $g(X \times \{v_j\})$ are disjoint for each i and j $(1 \leq i < j \leq 3)$. Suppose that for $i = 1$, 2, and 3, $g((E \cup F) \times \{v_i\})$ does not separate $g(X \times \{v_j\})$ from $g(X \times \{v_k\})$ in S^2 when $1 \leq j < k \leq 3$ and $j \neq i \neq k$. For $i = 1$ and 2, define H_i to be an arc in $S^2 - g((E \cup F) \times \{v_i\})$ that intersects both $g(X \times \{v_3\})$ and $g(X \times \{v_j\})$ $(1 \leq j \leq 2$ and $j \neq i)$.

Let z_1 and z_2 be points of A and B respectively. For $i = 1$, 2 and $j = 1$, 2, define M_{ij} to be

$$(g((E \cap C_j) \times \{v_i\}) \cap g((F \cap C_j) \times \{v_i\})) \cup (g(\{z_j\} \times R) \cap g((E \cup F) \times \{v_i\})).$$

Note that for $j = 1$ and 2, M_{1j} and M_{2j} are closed disjoint subsets of $g(C_j \times Y) - g(((E \cup F) - C_j) \times Y)$.

There exist mutually exclusive disks K_{11}, K_{12}, K_{21}, and K_{22} in S^2 such that for each i and j, the following conditions are satisfied:

1. The interior of K_{ij} contains M_{ij}.
2. K_{ij} does not intersect $H_i \cup g(((E \cup F) - C_j) \times Y) \cup g(X \times \{v_k, v_3\})$ when $1 \leq k \leq 2$ and $k \neq i$.

Let E_1, E_2, F_1, F_2, R_1, and R_2 be disjoint continua in $S^2 - g(X \times \{v_3\})$ that miss the interior of $\bigcup_{i,j=1}^2 K_{ij}$ such that for $n = 1$ and 2, E_n is in $g(E \times \{v_n\})$ and meets $\mathrm{Bd}\, K_{n1}$ and $\mathrm{Bd}\, K_{n2}$, F_n is in $g(F \times \{v_n\})$ and meets $\mathrm{Bd}\, K_{n1}$ and $\mathrm{Bd}\, K_{n2}$, and R_n is in $g(\{z_n\} \times R)$ and meets $\mathrm{Bd}\, K_{1n}$ and $\mathrm{Bd}\, K_{2n}$.

There exist arc-segments I_1, I_2, J_1, J_2, T_1, and T_2 in $S^2 - (g(X \times \{v_3\}) \cup \bigcup_{i,j=1}^2 K_{ij})$ whose closures are disjoint approximating E_1, E_2, F_1, F_2, R_1, and R_2, respectively, such that for $n = 1$ and 2, the following conditions are satisfied:

1. $\mathrm{Cl}\, I_n$ misses $H_n \cup g((F - (C_1 \cup C_2)) \times Y)$, meets $\mathrm{Bd}\, K_{n1}$ and $\mathrm{Bd}\, K_{n2}$, and contains a point e_n of $E_n - g((C_1 \cup C_2) \times \{v_n\})$.
2. $\mathrm{Cl}\, J_n$ misses $H_n \cup g((E - (C_1 \cup C_2)) \times Y)$, meets $\mathrm{Bd}\, K_{n1}$ and $\mathrm{Bd}\, K_{n2}$, and contains a point f_n of $F_n - g((C_1 \cup C_2) \times \{v_n\})$.

3. $\mathrm{Cl}\, T_n$ misses $g(((E \cup F) - C_n) \times Y)$ and meets Bd K_{1n} and Bd K_{2n}.

Let V be the complementary domain of $\bigcup_{i=1}^2 (I_i \cup J_i \cup T_i \cup \bigcup_{j=1}^2 K_{ij})$ that contains $g(X \times \{v_3\})$. Note that if i and j are distinct positive integers less than 3, then the continuum $g(X \times \{v_i, v_3\}) \cup H_j \cup K_{i1} \cup K_{i2} \cup I_i \cup J_i$ misses $K_{j1} \cup K_{j2} \cup I_j \cup J_j$. It follows that Bd V is a simple closed curve that contains T_1 and T_2 [6, Theorem 28, p. 156]. Consequently one of I_1, I_2, J_1, and J_2 does not meet Bd V. Suppose, without loss of generality, that $I_1 \cap \mathrm{Bd}\, V = \emptyset$. It follows that $\bigcup_{i=1}^2 (J_i \cup T_i \cup \bigcup_{j=1}^2 \mathrm{Bd}\, K_{ij})$ contains a simple closed curve L that separates e_1 from $g(X \times \{v_3\})$ in S^2. Let u be a point of $E - (C_1 \cup C_2)$ such that $g((u, v_1)) = e_1$. Since $g(\{u\} \times Y)$ is a continuum in $S^2 - L$ that meets e_1 and $g(X \times \{v_3\})$, we have a contradiction. Hence for some integer $i = 1, 2$, or 3, the continuum $g((E \cup F) \times \{v_i\})$ separates $g(X \times \{v_j\})$ from $g(X \times \{v_k\})$ in S^2 when $1 \le j < k \le 3$ and $j \neq i \neq k$.

Assume, without loss of generality, that $g((E \cup F) \times \{v_2\})$ separates $g(X \times \{v_1\})$ from $g(X \times \{v_3\})$ in S^2. This assumption contradicts the fact that $g(\{x\} \times Y)$ is a continuum in $S^2 - g((E \cup F) \times \{v_2\})$ that meets both $g(X \times \{v_1\})$ and $g(X \times \{v_3\})$. It follows that $X = E \cup F$.

Next we let h be a δ-map of $X \times Y$ onto D. Note that the set $h(E \times Y) \cap h(F \times Y)$ lies in $h((C_1 \cup C_2) \times Y)$ and meets both $h(C_1 \times Y)$ and $h(C_2 \times Y)$. Thus $h(E \times Y) \cap h(F \times Y)$ is not connected. But since $X = E \cup F$ and $h(X \times Y) = D$, the union of continua $h(E \times Y)$ and $h(F \times Y)$ is D, which contradicts the fact that D is unicoherent [6, Theorem 22, p. 175]. Hence X is hereditarily unicoherent.

Theorem 2. *If X is a λ connected hereditarily unicoherent continuum, then X is hereditarily decomposable.*

Proof. Assume that X contains an indecomposable continuum I. Let p and q be points of distinct composants of I [6, Theorem 139, p. 59]. Since X is λ connected, there exists a subcontinuum H of X that contains $\{p, q\}$ and does not contain I. But since p and q belong to different composants of I, the set $H \cap I$ is not connected, which contradicts the assumption that X is hereditarily unicoherent. Hence X is hereditarily decomposable.

Theorem 3. *Suppose that X and Y are continua, that X is λ connected, and that $X \times Y$ is disk-like. Then X is arc-like.*

Proof. By Theorem 1, X is atriodic and hereditarily unicoherent. Hence X is hereditarily decomposable (Theorem 2). It follows from Bing's theorem [1, Theorem 11] that X is arc-like.

Theorem 4. *Suppose that X and Y are λ connected continua. Then X and Y are arc-like if and only if $X \times Y$ is disk-like.*

Proof. Theorem 3 indicates that this condition is sufficient. To see that it is also necessary we note that if f is an $\epsilon/2$-map of X onto the unit interval $[0, 1]$ and g is an $\epsilon/2$-map of Y onto $[0, 1]$, then the function h of $X \times Y$ onto $[0, 1] \times [0, 1]$ defined by $h((x, y)) = (f(x), g(y))$ is an ϵ-map.

A continuum X is said to have the *fixed point property* if for each continuous function f of X into itself there is a point x of X such that $f(x) = x$. It is known [3] that every λ connected nonseparating plane continuum has the fixed point property.

Theorem 5. *If X and Y are λ connected continua and $X \times Y$ is disk-like, then X, Y, and $X \times Y$ have the fixed point property.*

Proof. O. H. Hamilton [5] proved that every arc-like continuum has the fixed point property. In [2] E. Dyer proved that all products of arc-like continua have the fixed point property. Hence the theorem follows from Theorem 4.

For another result involving products of λ connected continua see [4, Theorem 5].

Question 1. If X and Y are continua and $X \times Y$ is disk-like, then must X be arc-like?

Question 2. Does every disk-like continuum have the fixed point property?

An affirmative answer to Question 2 would imply that every nonseparating plane continuum has the fixed point property.

Added in proof. S. Mardešić asked Question 2 in [*Mappings of inverse systems*, Glasnik Mat.-Fiz. Astronom. 18 (1963), 241–254].

REFERENCES

1. R. H. Bing, *Snake-like continua*, Duke Math. J. 18 (1951), 653–663. MR 13, 265.

2. E. Dyer, *A fixed point theorem*, Proc. Amer. Math. Soc. 7 (1956), 662–672. MR 17, 1232.

3. C. L. Hagopian, *Another fixed point theorem for plane continua*, Proc. Amer. Math. Soc. 31 (1972), 627–628.

4. ———, *Planar λ connected continua*, Proc. Amer. Math. Soc. 39 (1973), 190–194. MR 47 #4230.

5. O. H. Hamilton, *A fixed point theorem for pseudo-arcs and certain other metric continua*, Proc. Amer. Math. Soc. 2 (1951), 173–174. MR 12, 627.

6. R. L. Moore, *Foundations of point set theory*, rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 13, Amer. Math. Soc., Providence, R. I., 1962. MR 27 #709.

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