PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 51, Number 2, September 1975

ON THE EXISTENCE OF TOTALLY INHOMOGENEOUS SPACES[1]

BRIAN M. SCOTT

ABSTRACT. A property of *total inhomogeneity* for topological spaces is defined and is shown to be stronger than that of *rigidity* (i.e., of having trivial autohomeomorphism group). It is further shown that compact, rigid, Hausdorff spaces are totally inhomogeneous, and that totally inhomogeneous spaces exist in profusion as dense subspaces of a class of locally compact spaces.

De Groot has studied in [3] the existence of spaces having specified autohomeomorphism groups, and in particular the existence of *rigid* spaces, i.e., spaces admitting only the identity autohomeomorphism. In this note we investigate the existence of spaces satisfying the stronger property of *total inhomogeneity* and the relation of such spaces to rigid spaces.

Definition. A space X is *totally inhomogeneous* iff given $x, y \in X$ with $x \neq y$, the spaces $X\setminus\{x\}$ and $X\setminus\{y\}$ are not homeomorphic.

Clearly, if X is totally inhomogeneous, then X is rigid. The following example shows that the converse implication fails, even for locally compact metric spaces.

Example. Let G be the directed graph of Figure 1; the automorphism group of G is isomorphic [2] to Z_3. Let P_4 be as in [3]. (We sketch the construction: Let D be the unit disk, and let $a_1, a_2, \ldots$ be a countable dense subset of the interior of D. Let V_1 be the open r_1-ball about a_1, where r_1 is chosen so that $V_1 \subseteq D$, and let F_1 be a 5-bladed "propeller" contained in V_1 and centered at a_1. Given r_i, V_i, F_i for $i < n$, let a'_n be the first term of the sequence $a_1, a_2, \ldots$ such that $a'_n \notin \bigcup\{F_i: i < n\}$. Let V_n be the open r_n-ball about a'_n, where $r_n < r_{n-1}/2$ is such that $V_n \subseteq D\setminus\bigcup\{F_i: i < n\}$, and let F_n be an $(n+4)$-bladed propeller contained in V_n

Received by the editors May 21, 1973 and, in revised form, April 3, 1974.
AMS (MOS) subject classifications (1970). Primary 54G15; Secondary 54B05, 54E45.
Key words and phrases. Rigid space, totally inhomgeneous space, locally compact space.
[1] During the preparation of this paper the author was a National Science Foundation Graduate Fellow.

and centered at a_n'. The space P_4 is $D\setminus\bigcup\{\mathrm{Int}(F_n): n = 1, 2, \ldots\}$, and is compact, connected, and rigid.) P_4 can be oriented and given "endpoints" by choosing $p_0, p_1 \in P_4 \cap \mathrm{Bdry}(D)$ on a diameter of D. Now replace each edge of G by a copy of P_4 in the appropriate orientation, and call the resulting space Y. The autohomeomorphism group of Y is [3] also isomorphic to Z_3. Form $X \subseteq Y$ by deleting the center of the 5-bladed propeller of the copy of P_4 replacing the edge v_0v_1 of G, and let x, y be the centers of the 5-bladed propellers of the copies of P_4 replacing v_1v_2 and v_2v_0. It is not hard to see that X is rigid, metric, and locally compact, but that $X\setminus\{x\}$ is homeomorphic to $X\setminus\{y\}$.

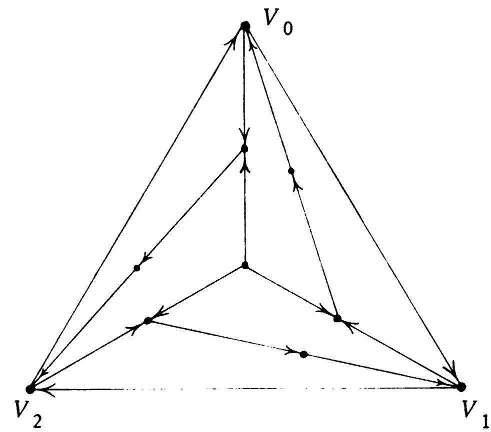

Figure 1

However, we do have

Theorem 1. *If X is a compact, rigid, Hausdorff space, then X is totally inhomogeneous.*

Proof. Suppose not; suppose $x \neq y$, but $h: X\setminus\{x\} \to X\setminus\{y\}$ is a homeomorphism. Let $\mathfrak{F} = \{V\setminus\{x\}: x \in V$ and V is open$\}$, and let $A = \bigcap\{\mathrm{cl}(h(F)): F \in \mathfrak{F}\}$. X is a compact and $h(\mathfrak{F})$ is a filterbase on X, so $A \neq \emptyset$. Let $a \neq y$, say $a = h(z)$ for some $z \in X\setminus\{x\}$. Pick disjoint open neighborhoods V and W of x and z, respectively, and let $F = V\setminus\{x\} \in \mathfrak{F}$; then $h(W)$ is open, and $h(W) \cap h(F) = \emptyset$, so $a \notin A$, and $A = \{y\}$. In fact, if V is any neighborhood of y, since $X\setminus V$ is compact, this argument shows that there are $F_1, \ldots, F_n \in \mathfrak{F}$ such that for $1 \leq i \leq n$, $\bigcap\{h(F_i)\} \subseteq V$; and $\bigcap\{h(F_i)\} = h(\bigcap\{F_i\}) \in h(\mathfrak{F})$, so $h(\mathfrak{F})$ converges to y. Hence [1] h extends to a 1-1, continuous map of X onto X which, X being compact Hausdorff, is a homeomorphism.

Totally inhomogeneous spaces exist in profusion. Guaranteeing this is

Theorem 2. *Let κ be an infinite cardinal, and let $\lambda = 2^{\kappa}$. Let $(X, \mathcal{U})$ be a compact Hausdorff space of cardinality λ such that if $V \in \mathcal{U}$ and $V \neq \emptyset$, then $|V| = \lambda$, and suppose that X has a basis of cardinality κ. Then there are subspaces R_α, $\alpha < \lambda$, each totally inhomogeneous, dense in X, and of cardinality λ, which are pairwise disjoint and pairwise nonhomeomorphic.*

Proof. Call $A \subseteq X$ a κ-G_δ (resp. κ-F_σ) set iff A is the intersection (resp. union) of κ open (resp. closed) sets in X. Let $\mathcal{C} = \{f: f$ is continuous, $\operatorname{dom} f \subseteq X$, and $\operatorname{ran} f \subseteq X\}$. Say $f \in \mathcal{C}$ is a *displacement* iff for some $A \subseteq \operatorname{dom} f$, $f(A) \cap A = \emptyset$ and $|f(A)| = \lambda$, and let $\mathcal{C}_0 = \{f \in \mathcal{C}: f$ is a displacement and $\operatorname{dom} f$ is a κ-G_δ set$\}$. Since there are only λ κ-G_δ subsets of X, each of which has a dense subset of cardinality κ, it follows that $|\mathcal{C}_0| = \lambda$. An immediate consequence of Theorem 2.2 of [4] is

Lemma 1. *If $f \in \mathcal{C}$ is a displacement, then there is $g \in \mathcal{C}_0$ such that $f = g \restriction \operatorname{dom} f$.*

We are now in a position to construct the R_α's.

Index $\mathcal{C}_0 = \{f_\alpha: \alpha < \lambda\}$, $\mathcal{K} = \{K \subseteq X: K$ is compact and $|K| = \lambda\} = \{K_\alpha: \alpha < \lambda\}$. Let $T = \{t_\alpha: \alpha < \lambda\}$ be an enumeration of $\lambda \times \lambda \times \lambda$. Finally, let $\mathcal{U} = \{U_\alpha: \alpha < \lambda\}$ enumerate $\mathcal{U}$.

If points, p_α, q_α, $k_\alpha \in X$ have already been chosen for $\alpha < \gamma < \lambda$, suppose that $t_\gamma = \langle \delta, \xi, \eta \rangle$, and choose p_γ, q_γ, $k_\gamma \in X$ distinct from all p_α, q_α, k_α for $\alpha < \gamma$ as follows: pick $p_\gamma \in \operatorname{dom} f_\delta$ such that $f_\delta(p_\gamma) \neq p_\gamma$ and, if possible, such that $p_\gamma \in U_\xi$ (if every point of $U_\xi \cap \{x \in \operatorname{dom} f_\delta : f_\delta(x) \neq x\}$ has already been chosen, we drop the requirement that $p_\gamma \in U_\xi$); let $q_\gamma = f_\delta(p_\gamma)$, and pick $k_\gamma \in K_\delta$ distinct from p_γ and q_γ. This is always possible for $\gamma < \lambda$, since $f_\delta \in \mathcal{C}_0$ and $|K_\delta| = \lambda$.

Now, for $\alpha < \lambda$, let $R_\alpha = \{p_\gamma : t_\gamma = \langle \delta, \eta, \alpha \rangle$ for some $\delta, \eta < \lambda\} \cup \{k_\gamma : t_\gamma = \langle \delta, \eta, \alpha \rangle$ for some $\delta, \eta < \lambda\}$; clearly $|R_\alpha| = \lambda$. Also, if $V \in \mathcal{U}$, then there is $W \in \mathcal{U}$ such that $\emptyset \neq W \subseteq \overline{W} \subseteq V$; and $\overline{W} \in \mathcal{K}$, so $|V \cap R_\alpha| \geq |\overline{W} \cap R_\alpha| = \lambda$ for any $\alpha < \lambda$. It follows immediately that the R_α are pairwise disjoint, dense subsets of X. To show that they are pairwise nonhomeomorphic and totally inhomogeneous, we actually prove more: if $V, W \in \mathcal{U}$ are such that $\emptyset \neq V \cap R_\alpha \neq W \cap R_\beta \neq \emptyset$, then $V \cap R_\alpha$ and $W \cap R_\beta$ are not homeomorphic. For if $h: V \cap R_\alpha \to W \cap R_\beta$ is a homeomorphism, choose $x \in (V \cap R_\alpha) \backslash (W \cap R_\beta)$ [if $V \cap R_\alpha \subseteq W \cap R_\beta$, consider h^{-1}, and choose $x \in$

$(W \cap R_\beta) \backslash (V \cap R_\alpha)]$; then $h(x) \neq x$, so there are disjoint $U, G \in \mathfrak{U}$ such that $x \in U \subseteq V$, $h(x) \in G \subseteq W$, and $h(U \cap R_\alpha) = G \cap R_\beta$. Note that $|G \cap R_\beta| = \lambda$, so h is a displacement, and for some $f_\gamma \in \mathcal{C}_0$, $h = f_\gamma \upharpoonright (V \cap R_\alpha)$. Also, for some $\delta < \lambda$, $U = U_\delta$. Let $\eta < \lambda$ be such that $t_\eta = \langle \gamma, \delta, \alpha \rangle$; then $|U \cap \{y \in \operatorname{dom} f_\gamma : f_\gamma(y) \neq y\}| = \lambda$, so $p_\eta \in R_\alpha \cap U_\delta \subseteq R_\alpha \cap V$, and $h(p_\eta) = f_\gamma(p_\eta) = q_\eta \notin R_\beta$, a contradiction.

Two corollaries to the proof of Theorem 2 are worth noting:

Corollary 1. *Theorem* 2 *still holds if "compact Hausdorff space" is replaced by "locally compact Hausdorff space"; the proof can be carried out in the one-point compactification.*

Corollary 2. *There are* 2^λ *totally inhomogeneous subspaces of* X, *each of cardinality* λ *and dense in* X, *which are pairwise nonhomeomorphic* (*but not pairwise disjoint*).

Proof. At each stage of the construction choose six points, p^i_α, q^i_α, k^i_α, for $\alpha < \lambda$ and $i = 0, 1$, such that if $t_\alpha = \langle \gamma, \delta, \eta \rangle$, then $q^i_\alpha = f_\gamma(p^i_\alpha) \neq p^i_\alpha$, $k^i_\alpha \in K_\gamma$, and, if possible, $p^i_\alpha \in U_\delta$. (As before, all points are chosen to be distinct.) Then for each $\phi: \lambda \to \{0, 1\}$, let $R_\phi = \{p^{\phi(\alpha)}_\alpha\} \cup \{k^{\phi(\alpha)}_\alpha\}$ $(\alpha < \lambda)$.

Say a space has property P if no two distinct, nonempty open subsets are homeomorphic: e.g., the R_α's constructed in the proof of Theorem 2 have property P. They are, however, very nonconstructive. Using far more constructive methods, Lozier [5] produces noncompact, 0-dimensional spaces X^κ of any infinite cardinality κ, in which any two distinct points have different values under a topologically invariant, ordinal-valued function which is unchanged by restriction to open subspaces: consequently, X^κ has property P. It is, however, not the case that total inhomogeneity implies property P, as is easily seen from the following

Example. Let X_0 be a closed line segment in E^2 of length l, and let x_0 be its midpoint. Form X_1 by attaching a segment of length $\frac{1}{2}$ to x_0, and let x_1, x_2, and x_3 be the midpoints of the "arms" of X_1. Form X_2 by attaching two segments to x_1, three to x_2, and four to x_3 in such a way that the new "arms" have length $\leq \frac{1}{4}$ and do not intersect X_1 except at x_1, x_2, and x_3. Continue in this fashion, and let $X = \bigcup \{X_n : n \in \omega\}$. Let Y be a copy of one of the components of $X \backslash \{x_0\}$, and let Z be the disjoint union of X and Y. Clearly Z does not have property P, and it is not hard to see that Z is totally inhomogeneous.

I wish to thank the referee for several valuable suggestions.

REFERENCES

1. J. Dugundji, *Topology*, Allyn and Bacon, Boston, Mass., 1966. MR 33 #1824.

2. R. Frucht, *Graphs of degree three with a given abstract group*, Canad. J. Math. 1 (1949), 365–378. MR 11, 377.

3. J. de Groot, *Groups represented by homeomorphism groups*. I, Math. Ann. 138 (1959), 80–102. MR 22 #9959.

4. J. de Groot and R. H. McDowell, *Autohomeomorphism groups of* 0-*dimensional spaces*, Compositio Math. 15 (1963), 203–209. MR 27 #4216.

5. F. W. Lozier, *A class of compact rigid* 0-*dimensional spaces*, Canad. J. Math. 21 (1969), 817–821. MR 39 #6243.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706

Current address: Department of Mathematics, Cleveland State University, Cleveland, Ohio 44115