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A COMMUTATIVE NOETHERIAN HOPF ALGEBRA OVER A FIELD IS FINITELY GENERATED

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ABSTRACT. Let k be an arbitrary field and H a commutative Hopf algebra over k. We give a short proof of the fact that H is Noetherian if and only if H is finitely generated as a k-algebra.

In [4] M. Takeuchi has shown that a commutative (or cocommutative) Hopf algebra H is faithfully flat over any sub-Hopf algebra K. He then uses this result to give a simple proof of the following:

(INJ) If H is a commutative Hopf algebra over a field k then the correspondence

$$\text{(sub-Hopf algebras)} \longrightarrow \text{(Hopf ideals)}$$
$$K \longmapsto K^+H$$

is injective.

As a corollary to (INJ) Takeuchi shows [4, 3.11] that any sub-Hopf algebra of a finitely generated, commutative Hopf algebra over a field is finitely generated. The purpose of this note is to show that a commutative Noetherian Hopf algebra over a field is finitely generated, and that this fact follows from (INJ) with essentially the same proof as Takeuchi's corollary.

For any Hopf algebra H over the field k, we let $H^+ = \ker(\epsilon)$ where ϵ is the counit of H, and we denote the antipode of H by S. A commutative Hopf algebra is said to be Noetherian (resp. finitely generated) if it is Noetherian (resp. finitely generated) as a k-algebra. For the basic facts about Hopf algebras see [1] or [3]. Clearly the desired result follows from the following

Proposition. *Let H be a commutative Hopf algebra over a field k. If H^+ is finitely generated as an ideal then H is finitely generated.*

Proof. Let $\{f_1, \ldots, f_n\}$ be generators of the ideal H^+. If C is the subcoalgebra of H generated by the f_i, then C is finite dimensional [1,

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1.4.3]. Since the antipode of H has order 2, it follows that $D = C + S(C)$ is a finite dimensional subcoalgebra of H which is stable under the antipode, and $\{f_1, \ldots, f_n\} \subset D$. If we let $[D]$ be the subalgebra of H generated by D, then, in fact, $[D]$ is a sub-Hopf algebra of H which is finitely generated.

Now $\{f_1, \ldots, f_n\} \subset [D] \cap H^+ = [D]^+$, and since the f_i are generators of the ideal H^+, we must have $H^+ \subset [D]^+H$. On the other hand, $[D]^+H$ is clearly contained in H^+. Thus $[D]^+H = H^+ = H^+H$, and so by (INJ) we must have $[D] = H$, which gives the desired conclusion.

One may easily generalize this result. For example, by using K. Newman's version of (INJ) for cocommutative Hopf algebras [2, 2.5][1] along with the proof of the Proposition (which essentially depends only on (INJ) and the fact that the antipode has finite order), one can show that a cocommutative Noetherian Hopf algebra over a field is finitely generated.

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[1] We thank the referee for pointing this out.