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THE FREENESS OF A GROUP BASED ON A DISTRIBUTIVE LATTICE[1]

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ABSTRACT. Let L be a distributive lattice and G the abelian group with the following presentation. The generators of G are the elements of the lattice L, and the relations are $(a \vee b) + (a \wedge b) = a + b$ where a and b are arbitrary elements of L. It is shown that G is free abelian. In particular, G is torsion free. The latter statement answers affirmatively a question posed several years ago by E. Weinberg.

This brief note is in the nature of an addendum to an earlier paper by one of the authors [2]. Our purpose is to prove that an abelian group based on a distributive lattice remains free under the relations $(a \vee b) + (a \wedge b) = a + b$. This result is basically a corollary of the two theorems in [2], but the application was previously overlooked. In fact, the second-named author initially proved our result, without reference to [2], using a theorem of G. Nöbeling [4]. However, it is the more recent results of [1] and [2] that yield the short proof that follows.

Theorem. *Let L be an arbitrary distributive lattice and let G be the abelian group generated by the elements of L with relations $(a \vee b) + (a \wedge b) = a + b$ for all $a, b \in L$. Then G is free.*

Proof. Denote by F the free abelian group based on the distributive lattice L (with no relations at all on F). Then F is isomorphic to the additive group of the semigroup ring $Z[S]$ where S is the semigroup associated with the set L and the operation $\wedge$ alone. The advantage of this identification is that F is now endowed, in a natural way, with multiplication and a ring structure. Indeed, F is a commutative ring generated by idempotents. We shall denote the additive structure of F by F^+ and, more generally, follow the notation of [2].

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Let R be the subgroup of F^+ generated by the elements $(x \vee y) + (x \wedge y) - x - y$ with x and y arbitrary elements of L. Using the distributive property of L, we observe that $z(x \vee y) = (z \wedge x) \vee (z \wedge y)$. Consequently, the equation

$$\begin{aligned} &z[(x \vee y) + (x \wedge y) - x - y] \\ &\qquad = (z \wedge x) \vee (z \wedge y) + (z \wedge x) \wedge (z \wedge y) - (z \wedge x) - (z \wedge y) \end{aligned}$$

is valid in the semigroup ring $F = Z[S]$ for all x, y and z in L, which implies that R is an ideal of F. Therefore, if G denotes $F^+/R^+ = (F/R)^+$, Theorem 2 in [2] implies that G is a direct sum of cyclic groups since F/R inherits from F the property of being a commutative ring generated by idempotents. In order to show that G is free and to complete the proof of the Theorem, it suffices to show that G is torsion free. Since F^+ is torsion free, it follows at once that $G\ (= F^+/R^+)$ is torsion free if and only if R^+ is pure in F^+. We now appeal to Theorem 1 in [2]. Since R is generated, as a ring, by the idempotents $(x \vee y) + (x \wedge y) - x - y$ in F, R^+ must be pure in F^+ according to Theorem 1, and the conclusion follows.

The following Corollary of the preceding Theorem settles a question of E. Weinberg posed in [5] and again in [3, p. 368] by J. Rotman; the motivation for the question is explained in [5].

Corollary. *The abelian group based on a distributive lattice with relations $a \vee b + a \wedge b = a + b$ is torsion free.*

The referee has kindly contributed the following improvement of the preceding result and has brought our attention to some recent related work of G.-C. Rota [6]. We express our gratitude to the referee for these contributions.

If DL, BR and CR denote the categories of distributive lattices, Boolean rings and commutative rings, respectively, we have the functors

$$CR \xrightarrow{U_1} BR \xrightarrow{U_2} DL,$$

where U_1 associates to a commutative ring its Boolean ring of idempotents and U_2 transforms a Boolean ring into a distributive lattice in the usual way. The functors U_1 and U_2 have left adjoints F_1 and F_2, respectively: F_2 adjoins zero and relative complements, and F_1 is discussed in §2 of [1]. Therefore, the composite $U_3 = U_2 U_1$ has a left adjoint, F_3, associating to a distributive lattice L the commutative ring C with generators $[x]$, $x \in L$, and relations

$$[x \wedge y] = [x][y], \qquad [x \vee y] = x + y - xy.$$

This is precisely the ring F/R described above. Since F_1 has free additive structures for its values [1] and since $F_3 = F_1 F_2$, the same must be true of F_3.

REFERENCES

1. G. M. Bergman, *Boolean rings of projection maps*, J. London Math. Soc. (2) 4 (1972), 593–598. MR 47 #93.

2. P. Hill, *The additive group of commutative rings generated by idempotents*, Proc. Amer. Math. Soc. 38 (1973), 499–502. MR 47 #4986.

3. J. M. Irwin and E. A. Walker (editors), *Topics in abelian groups* (Proc. Sympos., New Mexico State Univ., 1962), Scott, Foresman, Chicago, Ill., 1963. MR 29 #5888.

4. G. Nöbeling, *Verallgemeinerung eines Satzes von Herrn E. Specker*, Invent. Math. 6 (1968), 41–55. MR 38 #233.

5. E. Weinberg, *Free lattice-ordered abelian groups*, Math. Ann. 151 (1963), 187–199. MR 27 #3720.

6. G.-C. Rota, *The valuation ring of a distributive lattice*, Proc. Houston Lattice Theory Conference (to appear).

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