PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 51, Number 2, September 1975

A $W(Z_2)$ INVARIANT FOR ORIENTATION PRESERVING INVOLUTIONS

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ABSTRACT. In this paper we calculate an invariant in $W(Z_2)$, the Witt ring of nonsingular, symmetric Z_2-inner product spaces, for orientation-preserving involutions on compact, closed, connected $4n$-dimensional manifolds M. This invariant with the Atiyah-Singer index theorem uniquely determines the orthogonal representation of Z_2 on $H^{2n}(M; Z)/\mathrm{TOR}$. We also give an example to show that this invariant detects actions that the Atiyah-Singer theorem cannot.

In this paper we calculate a torsion invariant for orientation-preserving involutions. This invariant and the Atiyah-Singer-Segal G-signature theorem allow one to compute precisely the element of $W(\mathbf{Z}; \mathbf{Z}_2)$ given by the orthogonal representation of $\mathbf{Z}_2$ on $H^{2n}(M; \mathbf{Z})/\mathrm{TOR}$. This cannot be done with the Atiyah-Singer-Segal theorem alone. We will return to this in the last section.

In [4] Conner and Raymond define an invariant $q(T, M)$ for orientation-preserving actions of $\mathbf{Z}_p$, p a prime, T a generator of $\mathbf{Z}_p$, on closed, compact, oriented manifolds M of dimension $4n$. Briefly the action of $\mathbf{Z}_p$ on M gives us an orthogonal representation of $\mathbf{Z}_p$ on $H^{2n}(M; \mathbf{Q})$. If we denote by $w(T, M)$ and $\mathrm{sgn}(T, M)$ the rational Witt class and signature, respectively, of the inner product $(x, y) = p\langle x \cup y, [M]\rangle$ on the subspace of fixed vectors then $q(T, M) = w(T, M) - \mathrm{sgn}(T, M) \cdot 1$. In [1] the problem of expressing $q(T, M)$ in terms of fixed point information was solved for p odd. In this paper we compute $q(T, M)$ for $p = 2$ and give some applications.

First some notation and background. $W(R)$ will always denote the Witt group of nonsingular, symmetric, inner-product spaces over R, R a ring. $W(\mathbf{Z}_2)$ is isomorphic to $\mathbf{Z}_2$ and the isomorphism is given by taking the rank of the inner product space mod 2. From now on we identify $W(\mathbf{Z}_2)$ and $\mathbf{Z}_2$ by this isomorphism. If V is a disjoint union of a finite number of closed

Received by the editors June 11, 1974.
AMS (MOS) subject classifications (1970). Primary 57D85; Secondary 10C05.
Key words and phrases. Witt ring, peripheral invariant.

manifolds of possibly different dimensions, let $\chi_i(V)$ be the sum of the Euler characteristics of those components with dimensions equivalent to i mod 4.

Theorem. *For $p = 2$, $q(T, M) \equiv \chi_0(F)$ mod 2 where F is the fixed set of T.*

Since $\chi(M) \equiv \chi_0(F) + \chi_2(F)$ mod 2 and the Euler characteristic of any orientable manifold V^{4k+2} is even we get the following application of our theorem.

Corollary. *If $q(T, M) \not\equiv \chi(M)$ mod 2 then F has a nonorientable component of dimension equivalent to 2 mod 4.*

In §1 we give the necessary information on spectral sequences to prove the theorem in §2. §3 is devoted to a discussion of $W(\mathbf{Z}, \mathbf{Z}_2)$. I would like to thank Pierre Conner, Gary Hamrick, and James Vick for a number of helpful conversations.

1. In [4] it is shown that

$$q(T, M) = \sum_N w(T, N) - \operatorname{sgn}(T, N) \cdot 1 - \operatorname{per}(\partial N/T)$$

where N is a normal tube about a component of the fixed set F and $w(T, N)$ is the element of $W(\mathbf{Q})$ given by the obvious inner product on the image of $H^{2n}(N/T, \partial N/T; \mathbf{Q}) \to H^{2n}(N/T; \mathbf{Q})$ and $\operatorname{sgn}(T, N)$ is its signature. Since our action preserves orientation we have a bundle $\mathbf{R}P^{2k-1} \to \partial N/T \to F^{2l}$. Let Λ denote the local system of $\mathbf{Z}$-coefficients determined by $w_1(F)$, the first Stiefel-Whitney class. Applying [6] to this fibre bundle we get

Lemma 1. *There exists an E_1-spectral sequence whose E_2 term is*

$$E_2^{s,t} = \begin{cases} H^s(F; H^t(\mathbf{R}P^{2k-1})), & t \neq 2k-1, \\ H^s(F; \Lambda), & t = 2k-1. \end{cases}$$

There is also a homology spectral sequence whose E^2-term is

$$E^2_{s,t} = \begin{cases} H_s(F; H_t(\mathbf{R}P^{2k-1})), & t \neq 2k-1, \\ H_s(F; \Lambda), & t = 2k-1. \end{cases}$$

($\mathbf{Z}$-coefficients are assumed whenever the coefficients are not explicitly indicated.) In the homology spectral sequence $E^r_{2l,2k-1} = H_{2l}(F; \Lambda) \approx \mathbf{Z}$,

$r \geq 2$, and a generator is a permanent cycle. Denote a generator by $[\partial N/T]$.

Lemma 2. *$\cap[\partial N/T]: E_r^{s,t} \to E^r_{2l-s,2k-1-t}$, $r \geq 2$, is an isomorphism.*

Proof. This is proved in [7].

Because $\mathbf{R}P^{2k-1} \xrightarrow{i} \partial N/T \to F$ is covered by $S^{2k-1} \to \partial N \to F$, there exists a cohomology extension of the fibre $\theta: H^q(\mathbf{R}P^{2k-1}; \mathbf{Z}_2) \to H^q(\partial N/T; \mathbf{Z}_2)$ so that $i^* \circ \theta$ is the identity on $H^*(\mathbf{R}P^{2k-1}; \mathbf{Z}_2)$. The Leray-Hirsch theorem says

$$H^*(\partial N/T; \mathbf{Z}_2) \approx H^*(F; \mathbf{Z}_2) \otimes H^*(\mathbf{R}P^{2k-1}; \mathbf{Z}_2).$$

If we compare this with the E_2-term of our integral spectral sequence we get

Lemma 3. $E_\infty^{s,t} = E_2^{s,t}$, $0 < t < 2k - 1$.

2. We are now in a position to calculate the peripheral invariant on $\partial N/T$. Recall from [1] that this can be done by calculating the element of $W(\text{Fin})$ determined by the linking form on $\partial N/T$. Let $K = \{x \in H^{2n}(\partial N/T) | x = j^* y,\ y$ a torsion class in $H^{2n}(N/T)\}$ and $H = \{x \in H^{2n}(\partial N/T) | x = j^* y,\ x$ a torsion class$\}$. The following diagram will be useful.

$$\begin{tikzcd}
H^{2n}(N/T) \arrow[r, "j^*"] \arrow[d, "\cap[N/T]"'] & H^{2n}(\partial N/T) \arrow[r, "\delta"] \arrow[d, "\cap[\partial N/T]"'] & H^{2n+1}(N/T, \partial N/T) \arrow[d, "\cap[N/T]"'] \\
H_{2n}(N/T, \partial N/T) \arrow[r, "\partial"'] & H_{2n-1}(\partial N/T) \arrow[r, "j_*"'] & H_{2n-1}(N/T)
\end{tikzcd}$$

Lemma 4. *$x \in K^\perp = \{x | L(x, y) = 0$ for all $y \in K\}$ if and only if $x \cap [\partial N/T] \in \ker j_*$.*

Proof. $L(j^* y, x) = \langle \beta^{-1} y \cup \delta x, [N/T] \rangle = \langle \beta^{-1} y, \delta x \cap [N/T] \rangle$ where β is the Bockstein in the coefficient sequence $0 \to \mathbf{Z} \to \mathbf{Q} \to \mathbf{Q}/\mathbf{Z} \to 0$. There is also a nontrivial pairing of the torsion in $H^{2n}(N/T)$ and the torsion in $H_{2n-1}(N/T)$ given by $(x, \eta) \to \langle \beta^{-1} x, \eta \rangle$, $x \in H^{2n}(N/T)$, $\eta \in H_{2n-1}(N/T)$. These two facts show

$$x \in K^\perp \Leftrightarrow \delta x \cap [N/T] = 0 \Leftrightarrow x \cap [\partial N/T] \in \ker j_*. \quad \square$$

Now $\cap[\partial N/T]: E_\infty^{s,t} \to E^\infty_{2l-s,2k-1-t}$ is an isomorphism by Lemma 2 and the kernel of j_* is exactly the subgroup of $H_{2n-1}(\partial N/T)$ filtered by $\{E^\infty_{s,2n-1-s} | s < 2n - 1\}$. So $K^\perp$ is exactly the subgroup filtered by

$\{E_\infty^{2n-t,t} | t < 2k-1\}$. $K \subseteq K^\perp$ and we concentrate on $K^\perp/K$.

Lemma 5. *The form restricted to H/K is nonsingular and is equal to $w(T, N) - \mathrm{sgn}(T, N) \cdot 1$.*

Proof. In [1] it was shown how to compute the linking form on H/K from the inner-product on

$$L = \mathrm{Image}\{H^{2n}(N/T, \partial N/T)/\mathrm{TOR} \longrightarrow H^{2n}(N/T)/\mathrm{TOR}\}.$$

Using the transfer homomorphism as described in [3] it follows that

$$L^+ = \{x \in H^{2n}(N/T)/\mathrm{TOR} \mid rx \in L,\ r \in \mathbf{Z}\}$$

is isomorphic to $\mathrm{Hom}(L; \mathbf{Z})$. It now follows from [1] that the form on H/K is $w(T, N) - \mathrm{sgn}(T, N) \cdot 1$.

To finish our calculation of $\mathrm{per}(\partial N/T)$ it is only necessary to calculate the rank of $H^\perp/K$ which is isomorphic to $K^\perp/H$. This group is filtered by $E_\infty^{2n-t,t} \approx H^{2n-t}(F; H^t(\mathbf{R}P^{2k-1}))$ for $0 < t < 2k-1$. Since

$$\begin{aligned} H^{2n-t}(F; H^t(\mathbf{R}P^{2k-1})) &\approx H_{2l+t-2n}(F; H_{2k-t-1}(\mathbf{R}P^{2k-1})) \\ &\approx H^{2l+t-2n}(F; H^{2k-t}(\mathbf{R}P^{2k-1})) \end{aligned}$$

we have $\mathrm{rank}_{\mathbf{Z}_2} E_\infty^{2n-t,t} = \mathrm{rank}_{\mathbf{Z}_2} E^{2l+t-2n,2k-t}$. This shows the rank of $K^\perp/H$ is equal to the rank of $E_\infty^{2n-k,k}$ mod 2, but $E_\infty^{2n-k,k} = H^l(F; H^k(\mathbf{R}P^{2k-1}))$ and therefore

$$\mathrm{rank}\ K^\perp/H = \begin{cases} 0, & k \text{ is odd}, \\ \chi(F) \bmod 2, & k \text{ is even}. \end{cases}$$

This finishes the proof.

Proposition. $\mathrm{per}(\partial N/T) = w(T, N) - \mathrm{sgn}(T, N) \cdot 1 - \chi_0(F)$.

From this the Theorem in the introduction follows easily because

$$q(T, M) = \sum_i \{w(T, N_i) - \mathrm{sgn}(T, N_i) \cdot 1 - \mathrm{per}(\partial N_i/T)\}.$$

3. One consequence of this result has already been described in the introduction. Another application is that $q(T, M)$ and the Atiyah-Singer-Segal G-signature theorem completely determine the orthogonal representation of $\mathbf{Z}_2$ on $H^{2n}(M; \mathbf{Z})/\mathrm{TOR}$. $W(\mathbf{Z}; \mathbf{Z}_2)$ [2] is isomorphic to $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}_2$ as a group. If we introduce powers of ½ then any inner product module

splits into I, the fixed elements, and $I^\perp$ its orthogonal summand. We define

$$\operatorname{trs}(\langle T, V\rangle) = \langle I\rangle - \operatorname{sgn} I \cdot 1, \qquad \langle T, V\rangle \in W(\mathbf{Z}; \mathbf{Z}_2).$$

This is a torsion element in $W(\mathbf{Z}(\tfrac{1}{2}))$.

As a ring $W(\mathbf{Z}; \mathbf{Z}_2)$ can be described as follows. Let (m, n, t) be all triples in $\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}_2$ with $m \equiv n \bmod 2$ with the obvious addition and where multiplication is $(m_1, n_1, t_1) \cdot (m_2, n_2, t_2) = (m_1m_2, n_1n_2, n_1t_2 + n_2t_1)$. $W(\mathbf{Z}; \mathbf{Z}_2)$ is isomorphic to this ring by the map $(\langle T, V\rangle) \mapsto (\operatorname{sgn} V, \operatorname{sgn} I - \operatorname{sgn} I^\perp, \operatorname{trs}(\langle T, V\rangle))$. For an orientation preserving action of $\mathbf{Z}_2$ on M^{4n} the image of $(T, H^{2n}(M)/\mathrm{TOR})$ is $(\operatorname{sgn} M, \operatorname{sgn} F \cdot F, \operatorname{trs}(T, M))$ where $F \cdot F$ denotes the selfintersection of the fixed set. The first two coordinates are determined by the Atiyah-Singer-Segal theorem or the Hirzebruch theorem on involutions. The last coordinate is determined by $q(T, M)$ and the first two coordinates.

Proposition. $\operatorname{trs}(T, M) \equiv q(T, M) + \tfrac{1}{2}(\operatorname{sgn} M + \operatorname{sgn} F \cdot F) \bmod 2$.

Proof. The proof is a straightforward computation. If I represents the fixed elements in $H^{2n}(M; \mathbf{Z}(\tfrac{1}{2}))/\mathrm{TOR}$ then

$$\begin{aligned} q(T, M) = \langle 2\rangle\langle I\rangle - \operatorname{sgn} I \cdot 1 &= \langle 2\rangle \cdot (\langle I\rangle - \operatorname{sgn} I \cdot 1) + \operatorname{sgn} I(\langle 2\rangle - 1) \\ &= \langle 2\rangle \cdot \operatorname{trs}(T, M) \cdot \alpha + \operatorname{sgn} I \cdot \alpha, \end{aligned}$$

but $\langle 2\rangle \alpha = \alpha$ where α represents the form $\langle 2\rangle - 1$ whose image generates $W(\mathbf{Z}_2)$.

$$\operatorname{sgn} I = \tfrac{1}{2}(\operatorname{sgn} M + \operatorname{sgn} I - \operatorname{sgn} I^\perp) = \tfrac{1}{2}(\operatorname{sgn} M + \operatorname{sgn} F \cdot F)$$

by the Hirzebruch theorem. This completes the proof.

We now give an example that shows that $q(T, M)$ can detect actions that the Atiyah-Singer theorem cannot. Let $[r, \mathbf{C}P(2)]$ be $[Z_1, Z_2, Z_3] \mapsto [-Z_1, Z_2, Z_3]$: $[c; \mathbf{C}P(2)]$ be $[Z_1, Z_2, Z_3] \mapsto [\bar Z_1, \bar Z_2, \bar Z_3]$ and $[d; \mathbf{C}P(1) \times \mathbf{C}P(1)]$ the involution given by interchanging factors. Now consider $[r; \mathbf{C}P(2)] - [c; \mathbf{C}P(2)] - [d; \mathbf{C}P(1) \times \mathbf{C}P(1)]$. The action in the middle dimensional cohomology is given by

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

with respect to the obvious basis. The inner product is

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Obviously $\operatorname{sgn} M = 0$. A quick check shows $\operatorname{sgn} I - \operatorname{sgn} I^{\perp} = 0$ also but $\operatorname{trs}(T, M) = \alpha \neq 0$. The results in this section are due to P. Conner.

4. **One final comment.** The proof in §§1 and 2 can be generalized to p an odd prime to give the result for $q(T, M)$ announced in [1].

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