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A NOTE ON A BASIS PROBLEM

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ABSTRACT. It is shown that the functions $\{\exp -\lambda_\nu x\}_{\nu=1}^{\infty}$ form a basis for the subspace of $\mathfrak{L}_2(0, \infty)$ which they span if and only if

$$\inf_{\mu} \prod_{\nu=1;\nu\neq\mu}^{\infty} \left|\frac{\lambda_\nu - \lambda_\mu}{\bar{\lambda}_\nu + \lambda_\mu}\right| = \delta > 0.$$

The proof uses certain estimates concerning interpolation in H_2 due to Shapiro and Shields. The proof makes explicit a construction embedded in a paper of Binmore [1, Theorems 9–12].

1. **Introduction.** Suppose that $\{\lambda_\nu\}_{\nu=1}^{\infty}$ is a sequence of numbers in the right half-plane $x = \operatorname{Re} z > 0$. It is well known (see e.g. [2, p. 267]) that the sequence of functions $\{\exp -\lambda_\nu x\}_{\nu=1}^{\infty}$ is total in $\mathfrak{L}_2(0, \infty)$ if

$$\sum_{\nu=1}^{\infty} \operatorname{Re}\lambda_\nu(1 + |\lambda_\nu|^2)^{-1} = \infty, \ |\lambda_\nu| \to \infty,$$

and is nontotal in $\mathfrak{L}_2(0, \infty)$ if

$$\sum_{\nu=1}^{\infty} (1 + \operatorname{Re}\lambda_\nu)(1 + |\lambda_\nu|^2)^{-1} < \infty, \ |\lambda_\nu| \to \infty. \tag{1.1}$$

We denote by $\bar{V}$ the closure, in the $\mathfrak{L}_2$ norm, of the linear manifold spanned by the functions $\exp -\lambda_\nu x$, $\nu = 1, 2, 3, \ldots$. It was shown by Schwartz [5, p. 28] that if (1.1) is satisfied, so that $\bar{V} \neq \mathfrak{L}_2(0, \infty)$ then one may associete with each $f \in \bar{V}$ a unique expansion of the form

$$f(x) \sim \sum_{\nu=1}^{\infty} a_\nu \exp -\lambda_\nu x$$

We say that the functions $\exp -\lambda_\nu x$ form a basis for $\bar{V}$ if, for each $f \in \bar{V}$,

$$\lim_{n\to\infty} \left\| f(x) - \sum_{\nu=1}^{n} a_\nu \exp -\lambda_\nu x \right\| = 0.$$

It has been shown by Gurarii and Macaev [4] (see also [1, Theorem 10]) that, when the λ_ν's are all real, the functions $\exp -\lambda_\nu x$ are a basis for $\bar{V}$ if and

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only if $\lambda_{\nu+1}/\lambda_\nu > q > 1$, q independent of ν. Since their proof is rather complicated it seems worthwhile to show that their results, even in the case where the λ_ν's are complex, are a simple consequence of a theorem of Shapiro and Shields on interpolation in H_2 (Theorem B below). We shall prove

Theorem 1. *If* (1.1) *holds then a necessary and sufficient condition that the functions* $\{\exp -\lambda_\nu x\}_{\nu=1}^\infty$ *be a basis for the closed submanifold* $\overline{V}$ *of* $\mathfrak{L}_2(0, \infty)$ *which they span is that*

$$\inf_\mu \prod_{\nu=1;\nu\neq\mu}^{\infty} \left|\frac{\lambda_\nu - \lambda_\mu}{\overline{\lambda}_\nu + \lambda_\mu}\right| = \delta > 0. \tag{1.2}$$

Condition (1.1) is, of course, sufficient for the existence of the infinite Blaschke products appearing in (1.2).

2. **Reduction of the problem.** We assume acquaintance with Chapter 1 of [8]; in particular we shall use the notation of that chapter without further introduction. The proof of Theorem 1 follows from the following theorem which is part of [8, Theorem 7.1, pp. 58–59].

Theorem A. *Let* E *be a Banach space and* $\{x_n\}$ *a total sequence in* E *with* $x_n \neq 0$, $n = 1, 2, 3, \ldots$. *Then the following are equivalent*:

(a) $\{x_n\}$ *is a basis for* E.

(b) *There exists a constant* C *with* $1 \leq C < \infty$ *such that*

$$\left\|\sum_{i=1}^{n} \alpha_i x_i\right\| \leq C \left\|\sum_{i=1}^{m+n} \alpha_i x_i\right\| \tag{2.1}$$

for all positive integers m, n *and all complex numbers* $\alpha_1, \ldots, \alpha_{m+n}$.

(c)
$$\inf_{1<n<\infty} \operatorname{dist}(x_n/\|x_n\|, E^{(n)}) > 0 \tag{2.2a}$$

and

$$\inf_{1\leq n,k<\infty} \operatorname{dist}(\sigma_{(n)}, \sigma^{(n+k)}) > 0. \tag{2.2b}$$

The necessity of condition (1.2) follows immediately. It is well known (see e.g. [6, p. 98]) that the distance from the unit vector $(2 \operatorname{Re} \lambda_n)^{-1/2} \exp -\lambda_n x$ in $\mathfrak{L}_2(0, \infty)$ to the vector space spanned by the other functions is just

$$\delta(\lambda_n) = \prod_{\nu=1;\nu\neq n}^{\infty} \left|\frac{\lambda_\nu - \lambda_n}{\overline{\lambda}_\nu + \lambda_n}\right|,$$

and the necessity follows from (2.2a).

To prove that (1.2) is sufficient we use the Paley-Wiener isometry [3, p. 196]: if H_2 denotes the Hardy space of functions analytic for $x = \operatorname{Re} z > 0$ with the norm $\|F\|$ given by

$$\|F\|^2 = \frac{1}{2\pi} \sup_{x>0} \int_{-\infty}^{\infty} |F(x+it)|^2 \, dt$$

then $F \in H_2$ if and only if

$$F(z) = \int_0^{\infty} \exp(-zt) \overline{f(t)} \, dt,$$

for some $f(t) \in \mathfrak{L}_2(0, \infty)$ and, moreover $\|F\|_{H_2} = \|f\|_{\mathfrak{L}_2}$.

We suppose (1.2) to be satisfied and show that there is a constant $\beta = \beta(\delta)$ such that

$$\left\| \sum_{j=1}^{m+n} \alpha_j \exp -\lambda_j x \right\|_{\mathfrak{L}_2} \geq \beta \tag{2.3}$$

for every set $\{\alpha_j\}$ of complex numbers for which $\|\Sigma_{j=1}^{n} \alpha_j \exp -\lambda_j x\| = 1$. The sufficiency then follows from Theorem A (b). Now

$$\left\| \sum_{j=1}^{m+n} \alpha_j \exp -\lambda_j x \right\|_{\mathfrak{L}_2} = \|F\|_{H_2},$$

where

$$F(z) = \int_0^{\infty} \exp(-zt) \overline{\sum_{j=1}^{m+n} \alpha_j \exp -\lambda_j t} \, dt = \sum_{j=1}^{m+n} \frac{\overline{\alpha}_j}{z + \overline{\lambda}_j}.$$

Clearly the H_2 norm of $F(z)$ will not be altered by multiplying by the finite Blaschke product $\Pi_{j=1}^{m+n} (z + \overline{\lambda}_j)/(z - \lambda_j)$. This new function will be in the Hardy class H_2 of the left half-plane, but, by symmetry, we may replace $\overline{\lambda}_j$ by $-\lambda_j$, $j = 1, 2, 3, \ldots, m+n$, to obtain

$$\|F(z)\| = \|G(z)\|_{H_2}$$

where

$$G(z) = \sum_{j=1}^{m+n} \frac{\overline{\alpha}_j}{z + \overline{\lambda}_j} \left(\prod_{\nu=1; \nu \neq j}^{m+n} \frac{z - \lambda_\nu}{z + \overline{\lambda}_\nu} \right). \tag{2.4}$$

We write $G(z) = G_1(z) + G_2(z)$, where in $G_1(z)$ the sum in (2.4) is taken from $j = 1$ to n and $G_2(z)$ is the corresponding sum from $j = n+1$ to $m+n$, and show that if $\|G_1(z)\| = 1$ then $\|G(z)\| \geq \beta$ for some $\beta = \beta(\delta)$ assuming, of course, that (1.2) holds.

3. **Interpolation.** It is at this point that we make use of the results of Shapiro and Shields referred to previously. Their results are stated for the circle rather than for the half-plane.

Theorem B. *Suppose that the sequence* $\{\lambda_\nu\}$ *satisfies* (1.2). *Then there is a constant* $M(\delta)$ *such that*

(1) *the inequality*

$$\sum_{\nu=1}^{\infty} |g(\lambda_\nu)|^2(\operatorname{Re}\lambda_\nu) \le M(\delta)^2 \|g\|_{H_2}^2$$

holds for all $g \in H_2$;

(2) *if*

$$H(z) = \sum_{j=s}^{t} \frac{\bar{a}_j}{z+\bar{\lambda}_j}\left(\prod_{\nu=s;\nu\neq j}^{t} \frac{z-\lambda_\nu}{z+\bar{\lambda}_\nu}\right)$$

then

$$\|H(z)\|^2 \le M(\delta)^2 \sum_{j=s}^{t} |a_j|^2(\operatorname{Re}\lambda_j)^{-1}$$

for any s *and* t.

Theorem B is "cannibalized" from various theorems of [7]. Part (1) is [7, Theorem 1, Lemma 1] and part (2), dealing with Bessel sequences, is Theorem 3I on p. 525. We have also, for convenience, used $M(\delta)^2$ in place of $M(\delta)$.

We suppose firstly that $\sum_{j=1}^{m+n} |a_j|^2(\operatorname{Re}\lambda_j)^{-1} \ge K^2$, where K is some constant to be chosen later. In this case we apply Theorem B (1) to the function

$$g(z) = \sum_{j=1}^{m+n} \frac{\bar{a}_j}{z+\bar{\lambda}_j}\left[\prod_{\nu=1;\nu\neq j}^{\infty}\left(\frac{z-\lambda_\nu}{z+\bar{\lambda}_\nu}\theta_\nu\right)\right]$$

where the θ_ν are the constants of modulus 1 required to make the Blaschke products converge. We obtain

$$\sum_{j=1}^{m+n} \frac{|a_j|^2}{(2\operatorname{Re}\lambda_j)^2}(\operatorname{Re}\lambda_j)\left\{\prod_{\nu=1;\nu\neq j}^{\infty}\left|\frac{\lambda_j-\lambda_\nu}{\lambda_j+\bar{\lambda}_\nu}\right|\right\}^2 \le M(\delta)^2\|g(z)\|^2.$$

Now $\|g(z)\|^2 = \|G(z)\|^2$ and hence, using (1.2) again

$$\frac{\delta^2}{4}\sum_{j=1}^{m+n} |a_j|^2(\operatorname{Re}\lambda_j)^{-1} \le M(\delta)^2\|G\|^2,$$

from which

$$\|G\| \ge K\delta/2M(\delta).$$

If, secondly $\sum_{j=1}^{m+n} |a_j|^2(\operatorname{Re}\lambda_j)^{-1} < K^2$ then

$$\|G(z)\| \ge \|G_1(z)\| - \|G_2(z)\|$$

and, by Theorem B (2)

$$\|G_2(z)\|^2 \leq \frac{1}{2\pi} M(\delta)^2 \sum_{j=n+1}^{m+n} |\alpha_j|^2 (\operatorname{Re} \lambda_j)^{-1} \leq \frac{K^2 M(\delta)^2}{2\pi}.$$

Thus, in this case, since $\|G_1(z)\| = 1$ by hypothesis,

$$\|G(z)\| \geq 1 - KM(\delta)/(2\pi)^{1/2}.$$

The constant K is at our disposal and on choosing, for example, the optimal value

$$K = (M(\delta) + \delta/2M(\delta))^{-1}$$

we obtain

$$\|G(z)\| \geq \beta(\delta) = [1 + 2M(\delta)^2/\delta]^{-1}$$

and so Theorem 1 is proved.

Concluding remarks. It follows from Theorem 1 that, for our space $\overline{V}$, the condition (2.2b) in Theorem A is superfluous, since it is implied by condition (2.2a). It seems a question of some interest to determine for which Banach spaces E condition (2.2a) (for a total sequence $\{x_n\}$) is equivalent to $\{x_n\}$ being a basis for E.

In the case when (1.1) holds but (1.2) fails to hold it is an open question whether the manifold $\overline{V}$ has a basis at all. This could, presumably, depend on some special properties of the sequences $\{\lambda_\nu\}$ considered.

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