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THE METRIZABLE LINEAR EXTENSIONS OF METRIZABLE SETS IN TOPOLOGICAL LINEAR SPACES

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ABSTRACT. Suppose a subset X of a Hausdorff [locally convex] topological linear space (E, τ) is metrizable in its relative topology $\tau|X$. It is shown that if $\tau|X$ is separable, then there exists a metrizable [locally convex] linear topology τ_0 on the subspace V generated by X such that $\tau_0 \subset \tau|V$ and $\tau_0|X = \tau|X$ (Theorem 2). This is related to a recent result of Larman and Rogers which states that if, in addition, X is locally bounded, then τ_0 can be chosen to be normable (but then not necessarily $\tau_0 \subset \tau|V$) (Theorem 1). It is then observed that $\tau_0|X = \tau|X$ does not mean the coincidence of the corresponding induced uniformities on X. However, this is the case if the invariant uniformity compatible with τ is metrizable on X and X is convex (Theorem 4).

Notation. $E = (E, \tau)$ denotes a (real or complex) Hausdorff topological linear space, X a nonempty subset of E, V the linear subspace of E spanned by X.

X is said to be *locally bounded* if, for each x in X, there is a τ-neighbourhood C of the origin such that $(x + C) \cap X$ is bounded.

If λ is a topology on a space containing a set A, then $\lambda|A$ denotes the topology on A induced by λ.

Our purpose is to discuss the following theorem obtained recently by Larman and Rogers [4], and to simplify slightly its proof. Then we establish also a few related results.

Theorem 1. *Suppose E is locally convex, X locally bounded and $\tau|X$ second countable (= metrizable and separable). Then it is possible to introduce a norm $|\ |$ on V so that $\tau|X$ coincides with the relative topology of X as a subset of the normed space $(V, |\ |)$.*

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Actually, in the original formulation of Theorem 1, X is assumed to contain 0. That this condition is superfluous can be shown, apart from the proof of this theorem given below, as follows. Suppose Theorem 1 has been proved in the case $0 \in X$. Then, for an arbitrary X, choose any x_0 from X and consider $X_0 = X - x_0$. Let V_0 be the linear span of X_0. Since now $0 \in X_0$, there is a norm $|\ |_0$ on V_0 which induces $\tau|X_0$ on X_0. If $x_0 \in V_0$, then $V = V_0$ and we set $|\ | = |\ |_0$. If $x_0 \notin V_0$ then each $x \in V$ has a unique representation $x = v_0 + tx_0$, $v_0 \in V_0$, and we set $|x| = |v_0|_0 + |t|$. In both cases the norm $|\ |$ coincides with $|\ |_0$ on V_0, and since translations are homeomorphisms, the topologies on $X = X_0 + x_0$ induced by $|\ |$ and τ are identical.

As Professor C. A. Rogers explained in a letter dated November 20, 1973: "... A re-examination of the referee's example (see [4, p. 40]) shows that he did not actually prove the result we attributed to him. He actually disproved a stronger version of our Lemma 1 that we had originally used ...".

We should note that the proof of Theorem 1 given in [4] is valid only if E is real.

An analysis of the proof given by Larman and Rogers shows that the following three stages may be distinguished in it:

1°. Construction of a metrizable locally convex topology τ_1 on V such that $\tau_1 \subset \tau|V$.

2°. Construction of semimetrizable locally convex topology τ_2 on V such that $\tau_2 \subset \tau|V$ and $\tau_2|X = \tau|X$.

The topology $\tau_0 = \tau_1 \vee \tau_2$ on V is metrizable and locally convex, $\tau_0 \subset \tau|V$ and $\tau_0|X = \tau|X$. Then the last step is

3°. Construction of a norm required in Theorem 1 from a sequence of seminorms $|\ |_n$ defining τ_0.

Moreover, it can be easily observed that in 1° and 2° the local boundedness of X has not been used. This leads to

Theorem 2. *Suppose $\tau|X$ is second countable. Then $(V, \tau|V)$ is a continuous image of a metrizable separable topological space, so that $\tau|V$ is fully Lindelöf. Hence there is a metrizable linear topology τ_1 on V such that $\tau_1 \subset \tau|V$. Moreover, there exists a metrizable linear topology τ_0 on V such that $\tau_0 \subset \tau|V$ and $\tau_0|X = \tau|X$. If, in addition, τ is locally convex, then also τ_1 and τ_0 can be chosen to be locally convex.*

Proof. We can and do assume $E = V$. Let $\mathcal{C}$ be a base of open neighbourhoods of 0 in (E, τ), and $\mathcal{B}$ a countable base for $\tau|X$.

1°. To prove that $\tau|E$ is fully Lindelöf, we proceed as in the proof of Theorem V.1.1 in [2]. Let K denote the field of scalars of E, and M the *sum* of the metrizable separable spaces $K^n \times X^n$, $n \in N$. Then (E, τ) is the image of M under the continuous map f whose restriction to $K^n \times X^n$ is defined by

$$f((t_1, \ldots, t_n), (x_1, \ldots, x_n)) = \sum_{i=1}^{n} t_i x_i.$$

Since M is metrizable and separable, (E, τ) is fully Lindelöf. Since $E \setminus \{0\} = \bigcup\{E \setminus \overline{C}: C \in \mathcal{C}\}$, there is a countable subfamily $\mathcal{C}_1$ of $\mathcal{C}$ such that $\bigcap \mathcal{C}_1 = \{0\}$. We can assume that $\mathcal{C}_1$ is a base at 0 for a linear topology, τ_1, on E. Evidently, τ_1 is as required in the theorem.

The existence of τ_1 can be also proved in the folowing way (cf. [4]). If $A_1, \ldots, A_n$ are subsets of E, we define

$$W(A_1, \ldots, A_n) = \bigcup_{j=1}^{n} \cup \left\{\sum_{i=1}^{n} t_i A_i: |t_i| \leq 1 \ (i = 1, \ldots, n), \ |t_j| = 1\right\}.$$

For each finite sequence $B_1, \ldots, B_n$ in $\mathcal{B}$ such that the τ-closure of $W(B_1, \ldots, B_n)$ does not contain 0, we choose C in $\mathcal{C}$ so that $C \cap W(B_1, \ldots, B_n) = \emptyset$. Let $\mathcal{C}'$ be the countable subfamily of $\mathcal{C}$ obtained in this way. By adjoining to $\mathcal{C}'$ some other members of $\mathcal{C}$ (if necessary), we readily define a countable subfamily $\mathcal{C}_1$ of $\mathcal{C}$ which will be a base at 0 for a linear topology, τ_1, on E. Evidently τ_1 is semimetrizable and $\tau_1 \subset \tau$. We shall show in a moment that τ_1 is Hausdorff, and so metrizable.

Take any $x \neq 0$ from E. Then we can find linearly independent elements $x_1, \ldots, x_n$ in X such that

$$x = \sum_{i=1}^{n} s_i x_i = s \sum_{i=1}^{n} t_i x_i; \qquad s = \sup |s_i|, \qquad t_i = s_i/s.$$

Since x_i are linearly independent, there exists C in $\mathcal{C}$ such that $C \cap W(x_1 + C, \ldots, x_n + C) = \emptyset$. For each i the set $(x_i + C) \cap X$ is a neighbourhood of x_i in $(X, \tau|X)$, hence there exists B_i in $\mathcal{B}$ such that $x_i \in B_i \subset (x_i + C) \cap X$. It follows that

$$x/s \in W(B_1, \ldots, B_n) \subset W(x_1 + C, \ldots, x_n + C)$$

and $C \cap W(B_1, \ldots, B_n) = \emptyset$. Hence for some C_1 from $\mathcal{C}'$ we have $C_1 \cap W(B_1, \ldots, B_n) = \emptyset$, so that $(x/s) \notin C_1$. This proves that τ_1 is Hausdorff.

2°. We can suppose that each member of $\mathcal{B}$ can be written as $(x + C + C) \cap X$, , where x is taken from a countable subset of X and C from a countable subfamily $\mathcal{C}_2$ of $\mathcal{C}$. Clearly, we can also assume $\mathcal{C}_2$ to be a base at 0 for a linear topology, τ_2, on E. It is then obvious that τ_2 is semimetrizable, $\tau_2 \subset \tau$ and $\tau_2|X = \tau|X$.

Then the linear topology $\tau_0 = \tau_1 \vee \tau_2$ (for which the sets $C_1 \cap C_2$, $C_i \in \mathcal{C}_i$ $(i = 1, 2)$, form a base at 0) is as required in Theorem 2.

Proof of Theorem 1. In view of Theorem 2, there exists a sequence $|\ |_n$ of seminorms on V which defines a metrizable locally convex topology τ_0 on V such that $\tau_0 \subset \tau|V$ and $\tau_0|X = \tau|X$. Since X is locally bounded and $\tau|X$ is Lindelöf, we can represent X in the form $X = \bigcup_{n=1}^{\infty} X_n$, where each X_n is open in X, bounded in E, and $X_n \subset X_{n+1}$ $(n \in N)$. One can assume $|x|_n \leq 1$ for $x \in X_n$ $(n \in N)$. [Otherwise replace $|\ |_n$ by $m_n^{-1}|\ |_n$, $m_n = 1 + \sup\{|x|_n : x \in X_n\}$.] Then the formula

$$|x| = \sum_{n=1}^{\infty} 2^{-n}|x|_n$$

defines a norm on V. [For each $x \in V$, $|x| < \infty$. In fact, $x = \sum_{i=1}^{m} t_i x_i$, $x_i \in X$, and for k large enough all x_i are in X_k. Then for $n \geq k$ we have $|x|_n \leq \sum_{i=1}^{m} |t_i| = \text{const}$, hence the series defining $|x|$ converges.]

Let ν denote the topology on V determined by $|\ |$. Evidently $\tau_0 \subset \nu$. Let x_k, $x \in X$ and $x_k \to x$ in τ. Then $x \in X_m$ for some m and, since X_m is open in X, we can assume that all x_k are in X_m. It follows that

$$|x - x_k| \leq \sum_{n=1}^{r} |x - x_k|_n + 2^{-r+1}, \qquad r \geq m, \quad k \geq 1.$$

Since $|x - x_k|_n \to 0$ $(k \to \infty)$ for each n, it is now easy to conclude that $|x - x_k| \to 0$.

Thus $\nu|X \subset \tau|X = \tau_0|X$. Since $\tau_0 \subset \nu$, also $\tau_0|X \subset \nu|X$.

A similar construction of a norm which induces a given topology on each member of a sequence of bounded sets can be found in [1].

An analogue of Theorem 2 for groups sounds as follows.

Theorem 2′. *Suppose (G, γ) is a Hausdorff topological abelian group, A a subset of G, and H the subgroup generated by A. If $\gamma|A$ is second countable, then $(H, \gamma|H)$ is a continuous image of a metrizable separable space, hence $(H, \gamma|H)$ is fully Lindelöf, and there exists a metrizable group topology γ_0 on H such that $\gamma_0 \subset \gamma|H$ and $\gamma_0|A = \gamma|A$.*

The only major alteration that should be made in the proof of Theorem 2 is to use spaces $(-A)^m \times A^n$ in place of $K^n \times X^n$ [and sets of the form

$$W(t_1, \ldots, t_n; A_1, \ldots, A_n) = \sum_{i=1}^{n} t_i A_i,$$

where each t_i is either 1 or -1, instead of $W(A_1, \ldots, A_n)$].

Also, it is seen that part 2° of the proof of Theorem 2 can be easily modified to yield

Theorem 3. *Let Γ be a Hausdorff uniformity on a set X and γ the topology on X associated with Γ. If γ is second countable (=metrizable and separable), then there exists a metrizable uniformity Γ_0 on X which is coarser than Γ and compatible with γ.*

In the setting of Theorem 2, $\gamma = \tau_0|X = \tau|X$, and as Γ, Γ_0 we consider naturally the uniformities induced on X by the invariant uniformities compatible with τ and τ_0, respectively. We are going to show that neither "$\Gamma \neq \Gamma_0$" nor "Γ is nonmetrizable" can be excluded.

To this aim, consider the example given in [4, pp. 46–48]: $E = l^2$, τ is the weak topology $\sigma(l^2, l^2)$, and $X = \{ke_i: k \text{ is a nonzero integer}, i \in N\}$, where e_i denotes the ith unit vector of l^2. Then X is countable and $\tau|X$ is discrete, hence second countable, and as τ_0 we can choose the topology of coordinate-wise convergence in l^2. Let $x_n = n^2 e_n$ $(n \in N)$. Then $x_n \to 0$ in τ_0, hence the sequence (x_n) is τ_0-Cauchy. However, it is not τ-Cauchy because for $v = (1/n)_{n \in N}$ we have $\langle x_{2n} - x_n, v\rangle = n$, $n \in N$. Hence the identity mapping $(X, \Gamma_0) \to (X, \Gamma)$ is not uniformly continuous, and so Γ is not coarser than Γ_0.

Now suppose Γ has a countable base. This is equivalent to the assumption that the point 0 of the set $Y = X - X$ has a countable base of neighbourhoods in the space $(Y, \tau|Y)$. Therefore there is a sequence (v_n) in l^2 such that the sets

$$U_n = \{y \in Y: |\langle y, v_i\rangle| < 1;\ i = 1, \ldots, n\}$$

form a countable base at 0 in $(Y, \tau|Y)$. For each n, let m_n be such that

$$|\langle f_n, v_i\rangle| < 1/n \quad \text{for } i = 1, \ldots, n; \qquad f_n = e_{m_n}.$$

Clearly we can assume $m_1 < m_2 < \cdots$. Let $v = \sum_{n=1}^{\infty} n^{-1} f_n$. Then for each n, $nf_n \notin U$, where $U = \{y \in Y: |\langle y, v\rangle| < 1\}$. Hence none of U_n is contained in the neighbourhood U of 0 in $(Y, \tau|Y)$. It follows that Γ is not metrizable. Since all nf_n are in X, this argument shows also that $\tau|(X \cup \{0\})$ does not have a countable base at 0, hence it is not metrizable. It follows that in the assertion of Theorem 2, X cannot be replaced by its τ-closure.

If α is a linear topology, we denote by $\Gamma(\alpha)$ the (unique) uniformity compatible with α, and by $\Gamma(\alpha)|A$ the corresponding induced uniformity on a set A.

Theorem 4. *Suppose (E, τ) is a Hausdorff locally convex space over reals and X is a convex subset of E. Let A denote the closed absolutely convex hull of X and W the linear subspace of E generated by A. If $\Gamma(\tau)|X$ is metrizable, then there exists a metrizable locally convex topology μ on W such that $\mu \subset \tau|W$ and $\Gamma(\mu)|A = \Gamma(\tau)|A$.*

Proof. First observe that we can assume $0 \in X$. In fact, if $0 \notin X$, we can replace X by $X - x_0$ and argue a little subtler than on p. 324. Let $Y = X - X$. Since $\Gamma(\tau)|X$ is metrizable, there is a sequence $(U_n)_{n \in N}$ of absolutely convex open neighbourhoods of 0 in (E, τ) such that the sets $Y \cap U_n$ form a base at 0 in $(Y, \tau|Y)$. Since we can assume that $U_{n+1} + U_{n+1} \subset U_n$ $(n \in N)$, there exists a locally convex topology ν on E for which $(U_n)_{n \in N}$ is a base at 0. Evidently ν is semimetrizable and $\nu \subset \tau$. Since $X \subset Y$ and Y is absolutely convex, $A \subseteq \overline{Y}$. Also, since $\overline{Y}/2 \subset A$, it is clear that W is spanned by $\overline{Y}$. Now we shall show that the sets $\overline{Y} \cap U_n$ form a base at 0 in $(\overline{Y}, \tau|\overline{Y})$. Let U be any closed neighbourhood of 0 in (E, τ). Then there is U_n such that $Y \cap U_n \subset U$. Let $y \in \overline{Y} \cap U_n$. Then, given a neighbourhood V of 0 in (E, τ), $(y + V) \cap U_n$ is a τ-neighbourhood of y, hence there is x in $Y \cap U_n$ such that $x \in y + V$. It follows that $y \in (Y \cap U_n) + V \subset U + V$. Consequently, $y \in U$. Thus $\overline{Y} \cap U_n \subset U$.

Now it is quite obvious that the topology $\mu = \nu|W$ is Hausdorff, hence metrizable. By Grothendieck's lemma [3, 21.6(5)], both the identity mapping $(A, \Gamma(\tau)|A) \to (A, \Gamma(\mu)|A)$ and its inverse are uniformly continuous. Hence $\Gamma(\tau)|A = \Gamma(\mu)|A$. This completes the proof.

It is not clear to the author whether Theorem 4 is valid in complex spaces, as well as whether convexity of X is necessary. The assumption that X is convex can be omitted if X is compact (see [5, Theorem 5.2], and [4, Theorem 2]), (added in proof) or precompact (see [6, Theorem 1.4]).

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