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THE GENUS OF SUBFIELDS OF $K(n)$

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ABSTRACT. In this paper we fix a genus g and show that the number of fields of elliptic modular functions F of genus g is finite.

1. **Introduction.** Let Γ be the group of linear fractional transformations $w \to (aw+b)/(cw+d)$ of the upper half plane into itself with integer coefficients and determinant 1. Γ is isomorphic to the group of 2×2 matrices with integer entries and determinant 1 in which a matrix is identified with its negative. $\Gamma(n)$, the principal congruence subgroup of level n, is the subgroup of Γ consisting of those elements for which $a \equiv d \equiv 1 \pmod{n}$ and $b \equiv c \equiv 0 \pmod{n}$. G is called a congruence subgroup of level n if G contains $\Gamma(n)$ and n is the smallest such integer. G has a fundamental domain in the upper half plane which can be compactified to a Riemann surface and then the genus of G is defined to be the genus of the Riemann surface. We denote by $K(n)$ the field of elliptic modular functions of level n, i.e., the field of meromorphic functions on the Riemann surface corresponding to $\Gamma(n)$. If j is the absolute Weierstrass invariant, $K(n)$ is a finite Galois extension of $C(j)$ with $\Gamma/\Gamma(n)$ for Galois group. $\mathrm{SL}(2, n)$ is the special linear group of degree two with coefficients in Z/nZ and $\mathrm{LF}(2, n) = \mathrm{SL}(2, n)/\pm I$ where I is the identity matrix. Then $\Gamma/\Gamma(n) \cong \mathrm{LF}(2, n)$. If $\Gamma(n) \subseteq G \subseteq \Gamma$ and H is the corresponding subgroup of $\mathrm{LF}(2, n)$, then by Galois theory H corresponds to a subfield F of $K(n)$ and the genus of F, denoted by $g(F)$, equals the genus of G.

In this paper we fix a genus g and show that the number of F such that $C(j) \subseteq F \subseteq K(n)$ for some n amd such that $g(F) = g$ is finite. More precisely we prove that, for the fixed g, there are constants $r, t_1, \ldots, t_r$ such that any field of genus g is a subfield of $K(p_1^{t_1} \cdots p_r^{t_r})$ where $p_1, \ldots, p_r$ are the first r primes arranged in their natural order. A corollary to this result is a proof of a conjecture of H. Rademacher that the number of congruence subgroups of Γ of genus 0 is finite. Some previous results on the Rademacher

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conjecture have been obtained by Knopp and Newman [5], McQuillan [8] and the present author [1], [2]. The case of arbitrary genus g and $n = p^m$, a prime power, has been considered in [3]. The proof of the theorem is in two steps. First we show that there is an r such that any field of genus g is a subfield of $K(p_1^{x_1} \cdots p_r^{x_r})$ for some x_i, $1 \leq i \leq r$. Then we find constants $t_1, \ldots, t_r$ such that any field of genus g is a subfield of $K(p_1^{t_1} \cdots p_r^{t_r})$.

2. **Preliminaries.** The following notation will be standard. $G(L/K)$ is the Galois group of L over K. $g(K)$ is the genus of K. $K \cdot K'$ is the compositum of K and K' considered in some larger field containing both K and K'. $|A|$ denotes the order of the group A. $\langle c \rangle$ is the group generated by c. With the primes considered in their natural order, p_i is the ith prime. p_r is the largest prime p such that, for some x, $K(p^x)$ contains a field of genus $\leq g$ other than $C(j)$. p_r exists by [3, Proposition 2.6] and is larger than 3.

Suppose G is a subgroup of $G_1 \times G_2$. Let N_i = the projection of G onto G_i; $ft_1 = \{g_1 | g_1 \in G_1, (g_1, 1) \in G\}$; $ft_2 = \{g_2 | g_2 \in G_2, (1, g_2) \in G\}$. ft_i is called the ith foot of G. We will use extensively the following proposition on subgroups of the direct product of two finite groups which can be found in [7].

Proposition 1. *Suppose $G \subseteq G_1 \times G_2$ with G_1, G_2 finite. Then ft_i is a normal subgroup of N_i, $i = 1, 2$, and $N_1/ft_1 \cong N_2/ft_2$.*

We now collect some basic facts about the groups $\mathrm{LF}(2, m)$ which we will need. $|\mathrm{LF}(2, m)| = \frac{1}{2} m\phi(m)\psi(m)$ where $\phi(m)$ is the Euler ϕ function and $\psi(m) = m\Pi_{p|m}(1 + 1/p)$. Suppose p is a prime and consider the natural homomorphism f_r^n: $\mathrm{LF}(2, p^n) \rightarrow \mathrm{LF}(2, p^r)$ defined by reduction modulo p^r, $1 \leq r < n$. The kernel of $f_r^n = K_r^n$ and $|K_r^n| = p^{3(n-r)}$ if $p \neq 2$, $r \neq 1$; $|K_1^n| = 2^{3n-4}$ for $p = 2$. For $p > 3$, the only nontrivial normal subgroups of $\mathrm{LF}(2, p^n)$ are K_r^n, $1 \leq r < n$ [7]. The following lemma is proven in [4] for $p > 2$ and in [2] for $p = 2$.

Lemma 1. *If $|H \cap K_{n-1}^n| \leq p^2$, then $|H \cap K_t^n| \leq p^{2n-2t}$, $1 \leq t \leq n-1$.*

As an easy corollary to this we have

Corollary 1. *If H is a subgroup of K_t^n and $|H| \geq 2n - 2t + r$ for some r, $1 \leq r \leq n - t$, then $K_{n-r}^n \subseteq H$.*

The following is a collection of facts about fields and Galois groups which we will use. The proofs are straightforward and most can be found in a standard text such as Lang [6]. Suppose K and K' are subfields of L and $K \cap K' = k$.

(1) $G(L/K \cdot K') = G(L/K) \cap G(L/K')$.

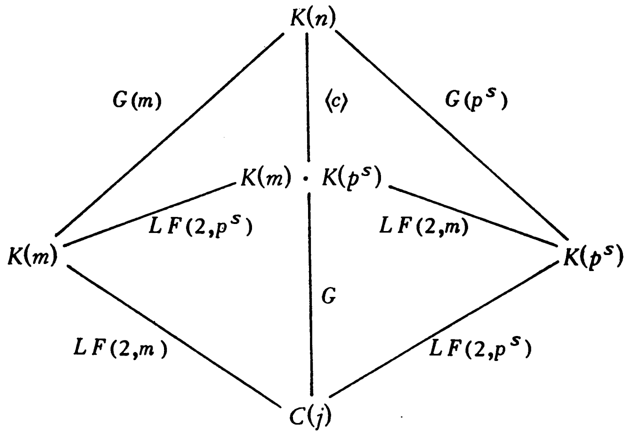
(2) $G(L/k) = G(L/K) \cdot G(L/K')$ if K or K' is normal over k.

(3) $G(K \cdot K'/k) \cong G(K/k) \times G(K'/k)$ with the isomorphism given by projecting σ in $G(K \cdot K'/k)$ onto both factors.

(4) $G(K \cdot K'/K) \cong G(K'/k)$ with the isomorphism given by restricting σ in $G(K \cdot K'/K)$ to K'.

(5) If $k \subseteq M \subseteq L$ and $k \subseteq F \subseteq K$ are fields with $L \cap K = k$, then in $K \cdot L$, $(F \cdot L) \cap (K \cdot M) = F \cdot M$.

3. **Main results.** Let $n = mp^s$ with $(p, m) = 1$ and p the largest prime dividing n. Consider the following diagram of fields and Galois groups.



$G \cong \mathrm{LF}(2, m) \times \mathrm{LF}(2, p^s)$. $G(m)$ is the kernel of the natural homomorphism from $\mathrm{LF}(2, n)$ to $\mathrm{LF}(2, m)$ and equals $\{\pm\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid a \equiv d \equiv 1,\ b \equiv c \equiv 0 \pmod{m}\}$. $\langle c \rangle$ has order 2 and is the kernel of the homomorphism from $\mathrm{LF}(2, n)$ to G. By the Chinese remainder theorem, $c = \pm\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$ with $a \equiv 1 \pmod{m}$ and $a \equiv -1 \pmod{p^s}$. Hence $\langle c \rangle$ is contained in the center of $\mathrm{LF}(2, n)$.

Lemma 2. $G(m) \cong \mathrm{SL}(2, p^s)$.

Proof. Consider θ: $\mathrm{SL}(2, p^s) \times \{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\} \longrightarrow \mathrm{LF}(2, m)$ given by:

$$\mathrm{SL}(2, p^s) \times \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\} \xrightarrow{i} \mathrm{SL}(2, p^s) \times \mathrm{SL}(2, m)$$

$$\xrightarrow{f} \mathrm{SL}(2, n) \xrightarrow{g} LF(2, n) \xrightarrow{h} LF(2, m)$$

where i is the injection, f is the isomorphism given by the Chinese remainder theorem, g is reduction mod $\pm I$ and h is the natural homomorphism. Then $G(m)$ equals the kernel of h and $g \circ f \circ i$ is 1-1 into $G(m)$ since the intersection of the kernel of g and the image of $f \circ i = I$. But $|G(m)| = p^s\phi(p^s)\psi(p^s) = |\mathrm{SL}(2, p^s)|$ so that the map is onto. Hence $\mathrm{SL}(2, p^s) \cong G(m)$.

Proposition 2. *Suppose $F \subseteq K(m)K(p^s)$ with $(m, p) = 1$, p the largest prime dividing n and $p > p_r$. If $g(F) \leq g$, then $F \subseteq K(m)$.*

Proof. Let $H = G(K(m) \cdot K(p^s)/F)$ so that $H \subseteq \mathrm{LF}(2, m) \times \mathrm{LF}(2, p^s)$. N_2, the projection of H onto $\mathrm{LF}(2, p^s)$, $= G(K(p^s)/F \cap K(p^s))$. But $g(F \cap K(p^s)) \leq g(F)$ and so by the assumption on p, $F \cap K(p^s) = C(j)$. Therefore $N_2 = \mathrm{LF}(2, p^s)$. ft_2 is normal in N_2 and, since $p > 3$, $ft_2 = K_t^s$ for some t. Therefore $N_2/ft_2 \cong \mathrm{LF}(2, p^t)$ and so p divides $|N_2/ft_2|$. But $N_2/ft_2 \cong N_1/ft_1$ so that p divides $|N_1|$. But $N_1 \subseteq \mathrm{LF}(2, m)$ and $p \nmid |\mathrm{LF}(2, m)|$. So $N_2 = ft_2$ and $N_1 = ft_1$. So $H = N_1 \times \mathrm{LF}(2, p^s)$ and by Galois theory, $F \subseteq K(m)$.

Proposition 3. *Suppose $F \subseteq K(mp^s)$ with $(m, p) = 1$, p the largest prime dividing n and $p > p_r$. If $g(F) = g$, then $F \subseteq K(m) \cdot K(p^s)$.*

Proof. Let $H = G(K(n)/F)$. If $c \in H$, we are done. So suppose $H \cap \langle c \rangle = I$. $H \cdot \langle c \rangle = G(K(n)/F \cap K(m)K(p^s))$. By Proposition 2, $F \cap K(m) \cdot K(p^s) \subseteq K(m)$ since $g(F \cap K(m) \cdot K(p^s)) \leq g(F)$. So $G(m) \subseteq H \cdot \langle c \rangle$. So

$$G(m) = G(m) \cap (H \cup cH) = (G(m) \cap H) \cup c \cdot (G(m) \cap H)$$

since $c \in G(m)$. Therefore $G(m) \cap H$ is a normal subgroup of index 2 in $G(m)$. But by Lemma 2, $G(m) \cong \mathrm{SL}(2, p^s)$ which has no subgroups of index 2 for $p > 3$ [7]. So $H \cap \langle c \rangle \neq I$.

Theorem 1. *If F has genus g, then $F \subseteq K(p_1^{x_1} \cdots p_r^{x_r})$ for some x_i, $1 \leq i \leq r$.*

Proof. Suppose $F \subseteq K(n)$ and p is the largest prime not in $\{p_1, \ldots, p_r\}$ which divides n. Write $n = mp^s$ with $(m, p) = 1$. Then by Proposition 3, $F \subseteq K(m)K(p^s)$ and then by Proposition 2, $F \subseteq K(m)$. Repeating the argument, one has, after a finite number of steps, $F \subseteq K(m)$ with $p_1, \ldots, p_r$ the only primes dividing m.

For $1 \leq i \leq r$, let e_i be the smallest power of p_i such that any field $\neq C(j)$ of genus $\leq g$ which is contained in $K(p_i^{x_i})$ for some x_i is actually contained in $K(p_i^{e_i})$ [3]. Suppose $p_i^{d_i} \| \prod_{j=i+1}^r (p_j^2 - 1)$. Since $K(p^x) \subseteq K(p^{x+1})$, we may assume in the following that, for all i, $x_i > e_i + d_i$.

Proposition 4. *Suppose $F \subseteq \prod_{i=1}^r K(p_i^{x_i})$ with $x_i > e_i + d_i$ and $g(F) \leq g$. Then*

$$F \subseteq K(p_1^{e_1+d_1}) \cdot K(p_2^{e_2+d_2+1}) \prod_{i=3}^{r} K(p_i^{e_i+d_i}).$$

Proof. The proof is by induction on the number of primes. Suppose

$$F \subseteq K(p_{r-1}^{x_{r-1}}) \cdot K(p_r^{x_r}) \quad \text{and} \quad H = G(K(p_{r-1}^{x_{r-1}}) \cdot K(p_r^{x_r})/F)$$

so that $H \subseteq \mathrm{LF}(2, p_{r-1}^{x_r}) \times \mathrm{LF}(2, p_r^{x_r})$. Then, since

$$N_2 = G(K(p_r^{x_r})/F \cap K(p_r^{x_r})) \quad \text{and} \quad (F \cap K(p_r^{x_r})) \subseteq K(p_r^{e_r}),$$

$N_2 \supseteq K_{e_r}^{x_r}$. There is an $H' \subseteq H$ such that $N_2' = K_{e_r}^{x_r}$. Then $|N_2'/ft_2'|$ divides p_r^y but $p_r \nmid |N_1'|$ since $N_1' \subseteq \mathrm{LF}(2, p_{r-1}^{e_r}).$ So $N_2' = ft_2' = K_{e_r}^{x_r}$. But $ft_2 \supseteq ft_2'$ so that $I \times K_{e_r}^{x_r} \subseteq H$ and $F \subseteq K(p_{r-1}^{x_r}) \cdot K(p_r^{e_r}) = L_1$. Similarly

$$N_1 = G(K(p_{r-1}^{x_{r-1}})/F \cap K(p_{r-1}^{x_{r-1}})) \quad \text{and so} \quad K_{e_{r-1}}^{x_{r-1}} \subseteq N_1.$$

There is an $H' \subseteq H$ such that $N_1' = K_{e_{r-1}}^{x_{r-1}}$. $|N_1'/ft_1'| = p_{r-1}^y$ and $N_1'/ft_1' \cong N_2'/ft_2'$. So $p_{r-1}^y | p_r^2 - 1$ and $y \leq d_{r-1}$. Let $|ft_1'| = p_{r-1}^z$. Then $(3x_{r-1} - 3e_{r-1}) - z = y < d_{r-1}$, i.e.

$$z > (3x_{r-1} - 3e_{r-1}) - d_{r-1} = (2x_{r-1} - 2e_{r-1}) + ((x_{r-1} - e_{r-1}) - d_{r-1})$$

and so, by the corollary to Lemma 1, $ft_1' \supseteq K_{e_{r-1}+d_{r-1}}^{x_{r-1}}$. So

$$K_{e_{r-1}+d_{r-1}}^{x_{r-1}} \times I \subseteq H \quad \text{and} \quad F \subseteq K(p_{r-1}^{e_{r-1}+d_{r-1}}) \cdot K(p_r^{x_r}) = L_2.$$

Then $F \subseteq L_1 \cap L_2$ which by fact (5) equals $K(p_{r-1}^{e_{r-1}+d_{r-1}})K(p_r^{e_r})$.

Now suppose

$$F \subseteq K(p_t^{x_t}) \cdot \prod_{i=t+1}^{r} K(p_i^{x_i}), \qquad F \cap \prod_{i=t+1}^{r} K(p_i^{x_i}) \subseteq \prod_{i=t+1}^{r} K(p_i^{e_i+d_i})$$

and

$$H = G\left(\prod_{i=t}^{r} K(p_i^{x_i})/F\right).$$

Then $N_2 \supseteq \Pi_{i=t+1}^r K_{e_i+d_i}^{x_i}$ and so there is an $H' \subseteq H$ such that $N_2' = \Pi_{i=t+1}^r K_{e_i+d_i}^{x_i}$. Then $N_2'/ft_2' \cong N_1'/ft_1'$, $|N_2'/ft_2'|$ divides $\Pi_{i=t+1}^r p_i^{y_i}$ and, if $p_{t+1} \neq 3$, no p_i divides $|N_1'|$. So

$$N_2' = ft_2' \quad \text{and} \quad ft_2 \supseteq ft_2' = \prod_{i=t+1}^{r} K_{e_i+d_i}^{x_i}.$$

So

$$F \subseteq K(p_t^{x_t}) \cdot \left(\prod_{i=t+1}^{r} K(p_i^{e_i+d_i})\right) = L_1.$$

If $p_{t+1} = 3$, then it is possible that $p_{t+1} \| \, |N_1'|$ in which case, arguing as in the 2nd part of the first step of the induction, one gets

$$F \subseteq K(p_t^{x_t}) \cdot K(p_{t+1}^{e_{t+1}+d_{t+1}+1}) \cdot \prod_{i=t+2}^{r} K(p_i^{e_i+d_i}) = L_1.$$

Similarly $K_{e_t}^{x_t} \subseteq N_1$ and so there is an $H' \subseteq H$ such that $N_1' = K_{e_t}^{x_t}$. Let $|N_1'/ft_1'| = p_t^y$ and $|ft_1'| = p_t^z$. Then, as before, $z > (2x_t - 2e_t) + ((x_t - e_t) - d_t)$ and so $ft_1' \supseteq K_{e_t+d_t}^{x_t}$. Therefore

$$F \subseteq K(p_t^{e_t+d_t}) \cdot \left(\prod_{i=t+1}^{r} K(p_i^{x_i})\right) = L_2.$$

Again $F \subseteq L_1 \cap L_2$ which equals $\Pi_{i=t}^{r} K(p_i^{e_i+d_i})$ unless $p_{t+1} = 3$ in which case case $e_{t+1} + d_{t+1}$ has to be replaced by $e_{t+1} + d_{t+1} + 1$.

Let $n = \Pi_{i=1}^{r} p_i^{x_i}$, $L = \Pi_{i=1}^{r} K(p_i^{x_i})$ and $A = G(K(n)/K(p_1^{x_1}p_2^{t_2} \cdots p_r^{t_r}))$ where $t_2 = e_2 + d_2 + 1$ and $t_i = e_i + d_i$, $i \neq 2$.

Proposition 5. *If $F \subseteq K(n)$ and $g(F) = g$, then $F \subseteq K(p_1^{x_1}p_2^{t_2} \cdots p_r^{t_r})$.*

Proof. Let

$$c_i = \pm\begin{pmatrix} a_i & 0 \\ 0 & a_i \end{pmatrix}, \ a_i \equiv 1 \left(\bmod \prod_{j=1;j\neq i}^{r} p_j^{x_j}\right), \ a_i \equiv -1 \pmod{p_i^{x_i}},$$

be the nontrivial element in the kernel of the homomorphism from LF$(2, n)$ to LF$(2, p_i^{x_i}) \times$ LF$(2, \Pi_{j=1;j\neq i}^{r} p_j^{x_j})$. Then C, the group generated by the c_i, $1 \leq i \leq r$, equals $G(K(n)/L)$, is contained in the center of LF$(2, n)$ and has order 2^{r-1}. $G(K(n)/F \cap L) = C \cdot H$ and $[CH: H] = 2^s$, $0 \leq s \leq r - 1$. By Proposition 4, $F \cap L \subseteq K(p_1^{x_1}) \cdot \Pi_{i=2}^{r} K(p_i^{t_i})$ and so

$$F \cap L \subseteq F \cap K(p_1^{x_1}p_2^{t_2} \cdots p_r^{t_r}).$$

Therefore

$$G(K(n)/K(p_1^{x_1}p_2^{t_2} \cdots p_r^{t_r}) \cap F) = A \cdot H \subseteq C \cdot H.$$

So we have $H \subseteq A \cdot H \subseteq C \cdot H$ and H is normal in $C \cdot H$ since C is in the center of LF$(2, n)$. So H is normal in AH and $AH/H \cong A/H \cap A$. So $H \cap A$ is a normal subgroup of A of index 2^t, $0 \leq t \leq s$. But $|A| = \Pi_{i=2}^{r} p_i^{3(x_i - t_i)}$ which is odd. So $A \cap H = A$ or $A \subseteq H$. Therefore $F \subseteq K(p_1^{x_1}p_2^{t_2} \cdots p_r^{t_r})$.

Proposition 6. *Suppose $F \subseteq K(n)$ with $n = 2^x m$, $(2, m) = 1$ and $g(F) = g$. Then $F \subseteq K(2^{t+1}m)$ where $t = e_1 + d_1$.*

Proof. As before, let

$$C = G(K(n)/K(2^x) \cdot K(m)) \quad \text{and} \quad A = G(K(n)/K(2^t m)).$$

$|C| = 2$. $F \cap K(2^x)K(m) \subseteq K(2^t)K(m)$ and so $F \cap K(2^x) \cdot K(m) \subseteq F \cap K(2^t m)$. Therefore $H \subseteq AH \subseteq CH$. Since $[CH: H] \leq 2$, there are 2 possibilities. If $H = C \cdot H$, then $H = A \cdot H$, $A \subseteq H$ and so $F \subseteq K(2^t m)$. If $[CH: H] = 2$ and $H = AH$, again $A \subseteq H$ and we are done. So assume $[AH: H] = 2$. Then since $AH/H \cong A/H \cap A$, $H \cap A$ is a normal subgroup of index 2 in A. Let

$$A' = G(K(2^x) \cdot K(m)/K(2^t)K(m))$$

and let $\phi: A \to A'$ be the homomorphism obtained by restricting an automorphism σ to $K(2^x) \cdot K(m)$. ϕ is an isomorphism and so $\phi(H \cap A)$ has index 2 in A'. But $A' = K_t^x$ and so

$$\phi(H \cap A) \supseteq K_{t+1}^x = G(K(2^x) \cdot K(m)/K(2^{t+1}) \cdot K(m)).$$

Therefore

$$G(K(n)/K(2^{t+1}m)) \subseteq H \cap A \subseteq H \quad \text{and} \quad F \subseteq K(2^{t+1}m).$$

Theorem 2. *If* $F \subseteq K(p_1^{x_1} \cdots p_r^{x_r})$ *has* $g(F) = g$, *then*

$$(*) \qquad F \subseteq K(p_1^{e_1+d_1+1} \cdot p_2^{e_2+d_2+1} \cdot p_3^{e_3+d_3} \cdots p_r^{e_r+d_r}).$$

Proof. Apply Propositions 5 and 6.

Combining Theorems 1 and 2, we obtain

Theorem 3. *Suppose* $F \subseteq K(n)$ *for some* n *and* $g(F) = g$. *Then* $(*)$ *holds.*

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