PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 51, Number 2, September 1975

AN INVARIANT IDEAL OF A GROUP RING OF A FINITE GROUP. II

J. S. HSIA AND ROGER D. PETERSON

ABSTRACT. The vanishing of the numerical invariant $\nu(G)$ of a finite group G is linked to the existence of certain central annihilators of the generic right ideal $\Gamma_R(G)$ in the group ring RG. This leads to several affirmative answers of questions posed in [1]. Also, some explicit values of $\nu(G)$ are described for the class of finite nonsolvable groups having all their odd Sylow subgroups cyclic.

For an associative ring R with identity and a finite group G, denote by $\Gamma_R(G)$ the right ideal of the group ring RG generated by elements of the form $\sigma(H) = \sum_{h \in H} h$, where H is a nontrivial subgroup of G. Since $g\sigma(H) = \sigma(gHg^{-1})g$ and $r\sigma(H) = \sigma(H)r$, we see that $\Gamma_R(G)$ is also the left ideal of RG generated by these $\sigma(H)$'s. An easy coset argument shows that the generators $\sigma(H)$ for $\Gamma_R(G)$ may be restricted to the prime-order subgroups of G. The invariant ideal $\gamma_R(G)$ of R with respect to G is then defined to be the (two-sided) ideal $R \cap \Gamma_R(G)$ [1]. When R is the ring of rational integers, we put $\Gamma(G) = \Gamma_Z(G)$, and $\gamma(G) = \gamma_Z(G) = \nu(G)\mathbf{Z}$ with $\nu(G) \geq 0$. This uniquely determines a nonnegative numerical invariant $\nu(G)$ for G; see also [3], [4]. In [1] three fundamental reduction theorems were proved, and using them, explicit theoretical as well as numerical results of the invariant for several large classes of groups were obtained. The present article continues the investigation for $\gamma_R(G)$. Recall a group is *tight* if it is a noncyclic group of order pq, where p and q are not necessarily distinct primes. In the solvable case, the numerical invariant is essentially determined by the numerical invariants of the tight subgroups (for instance, see [1, Theorem I.F.2]). This is generally not so in the nonsolvable case as SL$(2, p)$ has no tight subgroups for p a Fermat prime, and yet $0 \neq \nu(\mathrm{SL}(2, p))$ whenever $p \geq 17$, and also p divides $\nu(\mathrm{SL}(2, p))$. See [1, Proposition I.G.4 and Appendix IIIc]. For *nonsolvable* groups, the precise determination of $\nu(\mathrm{SL}(2, p))$ for p a Fer-

Received by the editors April 8, 1974.

AMS (MOS) subject classifications (1970). Primary 16A26, 20D99; Secondary 12A55, 20C05, 12B15.

Key words and phrases. Invariant ideal of a group ring, numerical invariant of a finite group, central annihilators, tight groups, condition (*), fixed-point-free representations.

mat prime is of paramount importance. Scharlau showed in [4] that $\nu(\mathrm{SL}(2, 5)) = 0$ because SL(2, 5) admits a fixed-point-free representation over the rationals. With the aid of a computer, thanks to the efforts of David Ford, we answer here affirmatively a question raised in [1]; namely: $\Gamma(\mathrm{SL}(2, 5))$ *has* a central annihilator α such that the coefficient $\mathrm{coeff}_\alpha(g_0)$ of α at g_0 is 1, for some $g_0 \in \mathrm{SL}(2, 5)$. From this result follow several interesting consequences. In particular, we have:

(i) For all R, $\gamma_R(\mathrm{SL}(2, 5)) = 0$;

(ii) $\nu(G_1 \times G_2) = \mathrm{g.c.d.}(\nu(G_1), \nu(G_2))$ if $\mathrm{g.c.d.}(|G_1|, |G_2|) = 1$.

Using theorems of Suzuki about the structures of groups all of whose odd Sylow subgroups are cyclic, we are able to determine in most cases the precise value for $\nu(G)$ where G is a nonsolvable group whose Sylow subgroups all have vanishing numerical invariants. Again, the exceptional cases occur when G contains a subgroup of the type $\mathrm{SL}(2, p)$, p a Fermat prime ≥ 17. We conclude this introductory discussion with two conjectures:

(A) $\nu(\mathrm{SL}(2, p)) = p$ for p a Fermat prime ≥ 17.

(B) If G is a solvable group and $\nu(G) = 1$, then G contains tight subgroups with differing numerical invariants.

Central annihilators. As was observed by Scharlau [4, Satz 1], the vanishing of the numerical invariant for G occurs precisely when G admits a fixed-point-free representation over the field of rationals. From the proof of this statement, it is clear that $\nu(G) = 0$ if and only if $\Gamma_{\mathbf{Q}}(G)$ has a nonzero central annihilator α in $\mathbf{Q}G$. On the other hand, the proof of [1, Theorem I.A] suggests that if $\nu(G) = 0$ then $\Gamma_R(G)$ should more-or-less have a central annihilator in RG with one of its coefficients equals to one. Indeed, we have

Theorem 1. *Suppose* $\nu(G) = 0$. *Then* G *has a normal subgroup* G_0 *such that*

(i) G_0 *contains every prime-order subgroup of* G, *and*

(ii) $\Gamma_R(G_0)$ *has a central annihilator* α *in* RG_0 *with* $\mathrm{coeff}_\alpha(g_0) = 1$ *for some* $g_0 \in G_0$.

We first need two lemmas and a definition. A finite group G is said to satisfy *condition* (*) if G has exactly one prime-order subgroup for each prime dividing the order $|G|$ of G.

Lemma 2. *Suppose* $\nu(G) = 0$. *Then* G *has a normal subgroup* H *such that* $[G: H] \leq 2$, *and* $H \cong L \times M$ *where* L *satisfies condition* (*), M *is either* 1, *or* SL(2, 3), *or* SL(2, 5), L *and* M *have relatively prime orders, and* H *contains every prime-order subgroup of* G.

Proof. The hypothesis $\nu(G) = 0$ implies all the odd Sylow subgroups of G are cyclic, and a Sylow 2-subgroup of G is either cyclic or generalized quaternion (see [1, I.B.2] or [4, Satz 3]). Suzuki's theorem [5, Theorem C] implies then the existence of an H with $[G\colon H] \leq 2$ with $H \cong L \times M$ where L is a group *all* of whose Sylow subgroups are cyclic, and M is either trivial or SL(2, p) for some prime p. As M cannot have any tight subgroups it must be either trivial or else p must be a Fermat prime (see [1, I.G.2]). And since $\nu(M)$ vanishes, p can only be either 3 or 5. L is solvable, so since $\nu(L)$ must also vanish, L satisfies condition (*) from the proof of Theorem I.F.2 (Case i) of [1]. As G has at most one involution, H clearly contains every prime-order subgroup of G.

Lemma 3. *If* K *satisfies condition* (*), *then there is a subgroup* H *which contains all the prime-order subgroups of* G, *and an element* α *in the center* $\mathfrak{Z}(RH)$ *of* RH, *for any* R, *such that* α *annihilates* $\Gamma_R(K)$ *and* $\mathrm{coeff}_\alpha(1) = 1$.

Proof. If $H_i = \langle h_i \rangle$, $1 \leq i \leq m$ are the distinct prime-order subgroups of K. Put $H = H_1 \times \cdots \times H_m$, and $\alpha = (1 - h_1) \cdots (1 - h_m)$. Then, α annihilates $\sigma(H_i)$ for every i, and so it annihilates $\Gamma_R(K)$. The coefficient of α at 1 is clearly 1.

Proof of Theorem 1. Since the center of $\mathbf{Z}G$ is contained in the center of RG and since $\Gamma_R(G)$ is the right ideal generated by elements of the form $\sigma(H)$, where H runs through the prime-order subgroups of G, it suffices to consider the case when $R = \mathbf{Z}$. As $\nu(G) = 0$, there exists a subgroup $H = L \times M$ with properties described in Lemma 2. Let N be the subgroup of L generated by its prime-order subgroups, so that N is cyclic of square-free order, and consider the subgroup $G_0 = N \times M$. Clearly, G_0 is normal in G and contains every prime-order subgroup of G. By Lemma 3, $\Gamma(N)$ has a central annihilator α in $\mathbf{Z}N$ such that $\mathrm{coeff}_\alpha(1) = 1$. If $M = 1$, we are done. If $M = \mathrm{SL}(2, 3)$, then the computations given in [1, Appendix IIIa] show that $\Gamma(M)$ has a central annihilator β in $\mathbf{Z}M$ with $\mathrm{coeff}_\beta(m_0) = 1$ for some $m_0 \in M$. With the aid of a computer (see [2]; also Appendix given below) the same result remains true for $M = \mathrm{SL}(2, 5)$. Thus, $\alpha\beta$ is a central element of $\mathbf{Z}G_0$, and it kills every prime-order subgroup P of G_0 since every prime-order subgroup of G_0 lies either in N or in M. Finally, $\mathrm{coeff}_{\alpha\beta}(m_0) = 1$.

Corollary 4. *If* $\nu(G) = 0$, *then* $\gamma_R(G) = 0$ *for all* R.

Proof. Let G_0 be a normal subgroup of G such that G_0 contains every prime-order subgroup of G and so $\Gamma_R(G_0)$ has a central annihilator α with $\operatorname{coeff}_\alpha(g_0) = 1$ for some $g_0 \in G_0$. By the first reduction theorem [1, Theorem I.D.2], $\gamma_R(G) = \gamma_R(G_0)$. On the other hand, if $r \in \gamma_R(G_0)$, then $r \cdot \alpha = 0$, so that $r = \operatorname{coeff}_{r\alpha}(g_0) = 0$. Thus, $\gamma_R(G) = \gamma_R(G_0) = 0$.

Corollary 5. *If H and K are groups of relatively prime orders, then $\gamma_R(H \times K) = \gamma_R(H) + \gamma_R(K)$. In particular, therefore, we have:* $\nu(H \times K) =$ g.c.d.$(\nu(H), \nu(K))$.

Proof. If $\nu(H) \neq 0$, then every prime divisor of $\nu(H)$ divides $|H|$ [4, Satz 5]. In particular, if $\nu(H)$ and $\nu(K)$ both do not vanish, then $\nu(H)R + \nu(K)R = R$, forcing $\gamma_R(H \times K) = R = \gamma_R(H) + \gamma_R(K)$. Hence, assume $\nu(H) = 0$. Corollary 4 yields $\gamma_R(H) = 0$. Also, H has a normal subgroup H_0 which contains every prime-order subgroup of H and such that $\Gamma_R(H_0)$ has a central annihilator α in RH_0 with $\operatorname{coeff}_\alpha(h_0) = 1$ for some $h_0 \in H_0$. Since $H_0 \times K$ contains every prime-order subgroup of $H \times K$, by the first reduction theorem [1, I.D.2], $\gamma_R(H \times K) = \gamma_R(H_0 \times K)$. Hence, we may assume $H = H_0$. Suppose $r \in \gamma_R(H \times K)$. Write

$$r = \sum_{h \in H} a_h \cdot h + \sum_{k \in K} b_k \cdot k,$$

where $a_h \in \Gamma_R(K)$ for all $h \in H$ and $b_k \in \Gamma_R(H)$ for all $k \in K$. Since α kills b_k for all $k \in K$ and is central in $R(H \times K)$, we have

$$r\alpha = \sum_{h \in H} a_h \cdot \alpha \cdot h.$$

Let $\alpha = \Sigma_{h \in H}\, c_h \cdot h$ with $c_h \in R$ for all $h \in H$. Then,

$$r\alpha = \sum_{y \in H} r \cdot c_y \cdot y = \sum_{y \in H} \left(\sum_{h \in H} a_h \cdot c_{h^{-1}y} \right) \cdot y.$$

But, $R(H \times K)$ is a free RK-module with basis H. Therefore,

$$r \cdot c_y = \sum_{h \in H} a_h \cdot c_{h^{-1}y}$$

belongs to $\Gamma_R(K)$ for all $y \in H$. As $c_y = 1$ for some $y \in H$, this gives $r \in \gamma_R(K)$. Hence we have:

$$\gamma_R(H \times K) = \gamma_R(K) = \gamma_R(H) + \gamma_R(K).$$

Corollary 6. *Suppose Q is a normal Sylow q-subgroup of G for some prime q. If $\nu(G/Q) = 0$, then we have:*

(i) $\nu(G) = 0$ or $\nu(G) = q$.

(ii) $\nu(G) = \text{g.c.d.}(\{\nu(T) | T$ *is a tight subgroup of* $G\})$.

(iii) $\gamma_R(G) = \nu(G)R$.

Proof. Let H be a complement of Q in G, and so $\nu(H) = 0$. As before, we may suppose $H = H_0$ and there is a central annihilator α in RH with $\text{coeff}_\alpha(h_0) = 1$ for some $h \in H$. By the third reduction theorem [1, Theorem I.D.4], we have $\gamma_R(G)$ is contained in $R \cap (\Gamma_R(H) + q \cdot RH)$. If r belongs to this intersection, write $r = a + qb$ with $a \in \Gamma_R(H)$ and $b \in RH$. Then, $r \cdot \alpha = \alpha \cdot a + q \cdot \alpha \cdot b = q \cdot \alpha \cdot b$. But then, $r = \text{coeff}_{r \cdot \alpha}(h_0) = \text{coeff}_{q \cdot \alpha \cdot b}(h_0) \in qR$. Hence, $\gamma_R(G) \subseteq qR$. In particular, if G has a tight subgroup with numerical invariant q, then we are finished. So, assume that G has no such tight subgroup. In particular, Q is cyclic or Q is generalized quaternion.

Case 1. Q is cyclic.

Let Q_0 be the unique subgroup of Q of order q. Then, $Q_0 \lhd G$, and the centralizer $C_G(Q_0)$ must contain every prime-order subgroup of G since otherwise G would have a nonabelian tight subgroup having numerical invariant q. But, $C_G(Q_0) = Q \times C_H(Q_0)$. Using the first two reduction theorems [1, I.D.2, I.D.3], we deduce:

$$\gamma_R(G) = \gamma_R(C_G(Q_0)) = \gamma_R(Q \times C_H(Q_0)) = \gamma_R(Q_0 \times C_H(Q_0)) = \gamma_R(C_H(Q_0)).$$

Clearly, $\nu(C_H(Q_0)) = 0$ since $\nu(H) = 0$. Thus, $\gamma_R(G) = 0$. Also, every tight subgroup of G lies in $C_H(Q_0)$ and so in H, but H has no tight subgroups.

Case 2. Q is a generalized quaternion.

G has a unique central involution. If $|Q| > 8$, then $G/C_G(Q)$ is a 2-group, so that $C_G(Q)$ contains every prime-order subgroup of G. But, $C_G(Q) = T \times C_H(Q)$, where T is the central subgroup of G of order two. This reverts back to Case 1 by the reduction theorems. Hence, we mav suppose that $|Q| = 8$. As proved in [1], either $G = Q \times H$ with H satisfying condition (*) and also being of odd order or else we have $G = \text{SL}(2, 3) \times H$ with H satisfying condition (*) and $6 \nmid |H|$. In either case, G is solvable and has no tight subgroups so that $\nu(G) = 0$. Thus we are done by [1, Theorem I.F.2].

Remark. Corollary 4 follows also from Corollary 6(iii) and the second reduction theorem. Indeed, if $\nu(G) = 0$, choose a prime q not dividing $|G|$. Consider $\tilde{G} = G \times Q$ where $Q \cong \mathbf{Z}/q\mathbf{Z}$. So, we have $\gamma_R(G) = \gamma_R(\tilde{G}) = \nu(\tilde{G})R = \nu(G)R = 0$.

Nonsolvable groups having cyclic odd Sylow subgroups. In the setting here $\nu(S)$ vanishes for all odd Sylow subgroups S of G. We shall consider two cases: (a) a Sylow 2-subgroup is generalized quaternion, and (b) a Sylow 2-subgroup is dihedral. In both these cases, Suzuki's structural theorems apply; see [5].

Case (a). G has subgroup G_0 of index at most two and G_0 is isomorphic to $H \times \mathrm{SL}(2, p)$, where H is a Zassenhaus group (i.e. every Sylow subgroup of H is cyclic) and p a prime ≥ 5, since G is nonsolvable. In particular, we observe that the orders of H and $\mathrm{SL}(2, p)$ are relatively prime. Also,as G has a unique involution, G_0 contains every prime-order subgroup of G so that $\gamma_R(G) = \gamma_R(G_0)$. Since H is solvable and all of its Sylow subgroups have vanishing numerical invariants, Theorem I.F.2 [1] can be applied to determine $\gamma_R(H)$. In particular, $\nu(H)$ is either 0, 1, or a prime dividing $|H|$. If p is *not* a Fermat prime, then $\nu(\mathrm{SL}(2, p)) = p$ (see [1, Corollary I.G.5]). Therefore, we deduce:

$$\nu(G) = \text{g.c.d.}(\nu(H), p) = \text{g.c.d.}(\{\nu(T) | T \text{ is a tight subgroup of } G\}).$$

If p is a Fermat prime and if $\nu(H) = 0$ (equivalently, H has no tight subgroups), then $\nu(G) = \nu(\mathrm{SL}(2, p))$ by the reduction theorems. Of course, when $\nu(H) = 1$, $\nu(G) = 1$ as well. On the other hand, if $\nu(H) = q$, then $q \nmid |\mathrm{SL}(2, p)|$. As $\nu(\mathrm{SL}(2, p)) \neq 0$ for $p > 5$, we conclude: $\nu(G) = 1$ for $p > 5$, and $\nu(G) = q$ when $p = 5$. Thus, if the precise value of $\nu(\mathrm{SL}(2, p))$ can be achieved for Fermat primes exceeding 5, the precise value for $\nu(G)$ can also be completely determined.

Case (b). Again, Suzuki's Theorem B [5] gives us the structure for G; namely, G contains a normal subgroup G_0 of index at most two, and $G_0 \cong H \times \mathrm{LF}(2, p)$ where H is a Zassenhaus group and $\mathrm{LF}(2, p)$ is the linear fractional group with $p \geq 5$. It is well known that $\mathrm{LF}(2, p) \cong \mathrm{PSL}(2, p)$ are simple groups, and so by [1, Corollary I.C.3] we have $\gamma_R(\mathrm{LF}(2, p)) = R$ for all R. Thus, $\gamma_R(G) = R$.

Remark. It is well known that a finite group all of whose abelian subgroups are cyclic is precisely a group which has each of its Sylow subgroups being either cyclic or generalized quaternion; in other words, precisely when all the Sylow subgroups have vanishing numerical invariants. A theorem of Artin-Tate then says such a group is exactly one that has periodic cohomology groups. Such groups are also of interest in topology where the groups operate on spheres fixed-point-freely.

If we wish to include the Klein 4-group as a dihedral group, then the discussion given in Case (b) remains valid. For, Suzuki's Theorem A [5] implies $G \cong H \times \mathrm{LF}(2, p)$ once again (for $p \geq 5$) and the simplicity of $\mathrm{LF}(2, p)$ yields $\gamma_R(G) = R$.

Appendix. We present here the summary results of a computer-assisted computation for the central annihilators of $\Gamma(\mathrm{SL}(2, 5))$. Denote by $\langle a, b, c, d\rangle$ the group element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and by $\mathrm{ccl}(y)$ the group ring element formed by summing the members in the conjugacy class of y.

There are nine conjugacy classes, and a $\mathbf{Z}$-module basis for the center $\mathfrak{Z}(\mathbf{Z}G)$ is given by:

$$\{R = 1 = \langle 1, 0, 0, 1\rangle, S = \langle -1, 0, 0, -1\rangle, T = \mathrm{ccl}(\langle 3, 2, 2, 0\rangle),$$

$$U = \mathrm{ccl}(\langle 2, 2, 2, 0\rangle), V = \mathrm{ccl}(\langle 3, 4, 1, 0\rangle), W = \mathrm{ccl}(\langle -1, -1, 1, 0\rangle),$$

$$X = \mathrm{ccl}(\langle 0, -1, 1, 0\rangle), Y = \mathrm{ccl}(\langle 1, -1, 1, 0\rangle), \text{ and } Z = \mathrm{ccl}(\langle 2, -1, 1, 0\rangle)\}.$$

If $\mathfrak{A}$ denotes the left annihilator ideal in $\mathbf{Z}G$ for $\Gamma(G)$, then a $\mathbf{Z}$-module basis for $\mathfrak{Z}(\mathbf{Z}G) \cap \mathfrak{A}$ is:

$$\{A = 2R - 2S + T - U - W + Y, \quad B = T - U - V + Z = (S - R)(U - Z)\}.$$

REFERENCES

1. J. S. Hsia and R. D. Peterson, *An invariant ideal of a group ring of a finite group, and applications*, J. Algebra 32 (1974), 576–599.

2. David Ford, J. S. Hsia and R. D. Peterson, *Some computations of the numerical invariant of a finite group* (in preparation).

3. M. Knebusch and W. Scharlau, *Über das Verhalten der Witt-Gruppe bei galoischen Körpererweiterungen*, Math. Ann. 193 (1971), 189–196. MR 45 #1955.

4. W. Scharlau, *Eine Invariante endlicher Gruppen*, Math. Z. 130 (1973). 291–296.

5. M. Suzuki, *On finite groups with cyclic Sylow subgroups for all odd primes*, Amer. J. Math. 77 (1955), 657–691. MR 17, 580.

DEPARTMENT OF MATHEMATICS, OHIO STATE UNIVERSITY, COLUMBUS, OHIO 43210 (Current address of J. S. Hsia)

Current address (R. D. Peterson): Department of Mathematics, University of Wisconsin, Milwaukee, Wisconsin 53201