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ON THE HEIGHT OF IDEALS GENERATED BY MATRICES

JOSEPH BECKER

ABSTRACT. A short geometric proof of the following algebraic theorem of Buchsbaum and Rim: Let R be the reduced local ring of an analytic variety and $g: R^t \to R^r$, $t \geq r$, be a homomorphism of R modules. Then the codimension of the support of the cokernel of $g \leq t - r + 1$.

In [2, Theorem 3.5], Buchsbaum and Rim prove the following

Theorem. *Let R be a noetherian ring, and $g: R^t \to R^r$, $t \geq r$, be a homomorphism of R modules. Then* ht $p = \dim R_p \leq t - r + 1$ *for all minimal primes p in* Supp(*cokernel of g*).

This result finds application to problems in algebraic geometry in [3]. However the proof given in [2] seems quite difficult, involving much preparation [1], and is highly algebraic in nature. In this short note I give a much easier proof for the case when R is the reduced local ring of an analytic variety. Unfortunately the proof does not seem to extend to arbitrary noetherian rings.

Let V be a complex analytic variety in $\mathbf{C}^m$ and $R = \mathcal{O}_V$, the sheaf of germs of holomorphic functions on V; then a R homomorphism $g: R^t \to R^r$ is just a holomorphic function $f: V \to \mathfrak{M}_{r\times t}(\mathbf{C})$, e.g., an element of $\mathfrak{M}_{r\times t}(\mathcal{O}_V)$, where $\mathfrak{M}_{r\times t}$ is just the r by t matrices. In the above correspondence it is easily seen by the theory of determinants over a commutative ring that g is onto if and only if some r by r det of the matrix is a unit in R. Hence $\{x \in V: \operatorname{rank}_x f < r\} =$ supp(coker g) where supp(coker g) denotes $\{x: \operatorname{coker}_x g \neq 0\}$ which is just locus(I) where I is the annihilator of coker g. Thus $\operatorname{codim}_V\{x \in V: \operatorname{rk}_x f < r\} = \operatorname{codim}_V$ locus $I = \operatorname{codim}_V$ locus $p =$ ht p for all minimal primes containing I, where by codim_V we mean the maximum codimension over all points in the set.

Now $\mathfrak{M}_{r\times t}(\mathbf{C}) = \mathbf{C}^{rt}$ and the subset A of matrices of nonmaximal rank

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form an algebraic subvariety (zeroes of dets) whose codim is well known to be $t - r + 1$. We include a proof of this for the sake of completeness. Now $A_{r-1} = \{a \in A: \operatorname{rk} a = r - 1\}$ is an open dense subset of A so it suffices to compute codim A_{r-1}. Given $a \in A_{r-1}$, there exist $r - 1$ columns of a (which by change of basis we assume are the first $r - 1$ columns) that are linearly independent. Let $c_1, \ldots, c_t$ denote the columns of a and $h_i(a) = [c_1, c_2, \ldots, c_{r-1}, c_i]$, $i = r, \ldots, t$, be $r \times r$ matrices. If each det $h_i(a) = 0$, then all r by r dets of a vanish so A_{r-1} is defined by $t - r + 1$ equations. (Because det $h_i(a) = 0$ implies there exists a nontrivial relation, $b_{ii} \neq 0$, $\sum_{j=1}^{r-1} b_{ij}c_j + b_{ii}c_i = 0$, if some other $\det[c_{k_1}, \ldots, c_{k_r}] \neq 0$ then $\det[b_{k_1k_1}c_{k_1}, \ldots, b_{k_rk_r}c_{k_r}] \neq 0$, but this last matrix is generated by only $r - 1$ columns.)

Now let W = locus I, $l = \dim V - \dim W = \operatorname{ht} I$ and $p \in W$. It is a standard result in local analytic geometry [5] that there is an affine linear subspace T of the ambient space such that $\dim_z T \cap W = 0$ and $\dim_z T \cap V = l$. Let $Z = T \cap V$. Note that $f(p) \in A$ and $f(Z - p) \cap A = \emptyset$. Since f has discrete fibers ($\dim f^{-1}f(p) = \dim_p T \cap W = 0$) on Z, $f(Z)$ is locally an l dimensional analytic set. Another standard result in analytic geometry is that if B_1, B_2 are analytic varieties then $\dim B_1 \cap B_2 \geq \dim B_1 - \operatorname{codim} B_2$ provided $B_1 \cap B_2 \neq \emptyset$. Now $f(Z) \cap A$ = a point so

$$0 = \dim f(Z) \cap A \geq \dim f(Z) - \operatorname{codim} A = l - (t - r + 1)$$

i.e. $t - r + 1 \geq l$.

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DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907