PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 51, Number 2, September 1975

RESTRICTIONS OF ANALYTIC FUNCTIONS. II

MARVIN ROSENBLUM AND JAMES ROVNYAK[1]

ABSTRACT. An isometric expansion is derived which recaptures any H^2 function from a restriction of its boundary function to a Borel set.

1. **Introduction.** Let Δ be a Borel subset of the real line such that neither Δ nor its complement Δ^c is a Lebesgue null set. Let $f(x)$ be a complex valued measurable function on Δ. In [4] we derived conditions for the existence of a function $F(z)$ in H^2 whose boundary function $F(x)$ agrees with $f(x)$ a.e. on Δ. General methods for recapturing $F(z)$ from a knowledge of $f(x)$ have been given by Golusin and Krylov [1] and Patil [3]. When Δ is an interval, say $\Delta = (0, \infty)$, there is a more refined theory due to van Winter [9] which shows how $F(z)$ can be recaptured from $f(x)$ by means of reciprocal formulas of the Mellin form. Closely related results were obtained independently by Kreĭn and Nudel′man [2]. See also Steiner [7].

We give a new derivation of the Mellin representation of H^2. Our main purpose, however, is to extend the representation to the case where Δ is a general Borel set. The proof uses Cayley inner functions and the methods of [5] to reduce the general result to the special case where Δ is an interval.

2. **Mellin representation of H^2 functions.** By H^2 we mean the space of functions $F(z)$ analytic for $y > 0$ such that

$$\sup_{y>0} \int_{-\infty}^{+\infty} |F(x+iy)|^2\, dx < \infty.$$

Theorem 1 (van Winter [9]). (i) *If $\mathfrak{F}(t)$ is a measurable function on $(-\infty, \infty)$ such that*

Received by the editors June 12, 1974.

AMS (MOS) subject classifications (1970). Primary 30A78, 30A88; Secondary 42A76, 47B99.

Key words and phrases. H^2, Cayley inner function, boundary function, Mellin transform.

[1] This research was supported by NSF Grant GP-31483X. The first author also received support from the Center for Advanced Studies of the University of Virginia.

$$\int_{-\infty}^{+\infty} (1 + e^{2\pi t})|\mathfrak{F}(t)|^2 \, dt < \infty, \tag{1}$$

then

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} z^{-\frac{1}{2}+it}\mathfrak{F}(t)\, dt, \qquad 0 < \arg z < \pi, \tag{2}$$

defines a function in H^2 *whose boundary function* $F(x) = F(x + i0)$ *satisfies*

$$\int_0^{\infty} |F(x)|^2 \, dx = \int_{-\infty}^{+\infty} |\mathfrak{F}(t)|^2 \, dt, \tag{3}$$

$$\int_{-\infty}^{0} |F(x)|^2 \, dx = \int_{-\infty}^{+\infty} e^{2\pi t}|\mathfrak{F}(t)|^2 \, dt, \tag{4}$$

and indeed, more generally,

$$\int_0^{+\infty} |F(re^{i\theta})|^2 \, dr = \int_{-\infty}^{+\infty} e^{2\theta t}|\mathfrak{F}(t)|^2 \, dt \tag{5}$$

for each fixed θ, $0 \leq \theta \leq \pi$. *Conversely, if* $F(z)$ *is an* H^2 *function, there exists an essentially unique function* $\mathfrak{F}(t)$ *satisfying* (1) *such that* (2)–(5) *hold. The inversion formula*

$$\mathfrak{F}(t) = \lim_{T\to\infty} \frac{1}{\sqrt{2\pi}} \int_{1/T}^{T} (re^{i\theta})^{-\frac{1}{2}+it} F(re^{i\theta}) e^{i\theta} \, dr \tag{6}$$

holds for each fixed θ, $0 \leq \theta \leq \pi$, *where convergence is in the metric of* $L^2(-\infty, \infty)$.

(ii) *Suppose* $F(x) \in L^2(0, \infty)$, *and define*

$$\mathfrak{F}(t) = \lim_{T\to\infty} \frac{1}{\sqrt{2\pi}} \int_{1/T}^{T} x^{-\frac{1}{2}+it} F(x) \, dx \tag{7}$$

with convergence in the metric of $L^2(-\infty, \infty)$. *Then there exists a function* $F(z)$ *in* H^2 *such that* $F(x + i0) = F(x)$ *a.e. on* $(0, \infty)$ *if and only if* $\mathfrak{F}(t)$ *satisfies* (1), *and in this case* $F(z)$ *satisfies* (2)–(6).

Proof. The space H^2, regarded as a Hilbert space in the usual norm, has reproducing kernel $K_0(w, z) = (2\pi i)^{-1}(w^* - z)^{-1}$. Let $\mathcal{K}$ be the Hilbert space of functions

$$G(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{tz}\mathfrak{F}(t)\, dt, \qquad 0 < \operatorname{Re} z < \pi, \tag{8}$$

where $\mathfrak{F}(t)$ satisfies (1) and $\|G\|^2$ is equal to the expression in (1). Routine arguments show that $\mathcal{K}$ has reproducing kernel

$$K(w, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{tz} e^{tw^*} (1 + e^{2\pi t})^{-1} \, dt.$$

Since by [8, p. 192]

$$\int_0^\infty x^{s-1}(1+x)^{-1}\,dx = \pi\csc(\pi s), \qquad 0 < \operatorname{Re} s < 1,$$

we obtain

$$K(w, z) = e^{iz/2}K_0(e^{iw}, e^{iz})e^{-iw^*/2}.$$

It follows that $F(z) \longrightarrow e^{iz/2}F(e^{iz})$ is an isometry mapping H^2 onto $\mathcal{K}$ with inverse $G(z) \longrightarrow z^{-1/2}G(-i\log z)$, $0 < \arg z < \pi$. Therefore H^2 is exactly the set of functions in the upper half-plane of the form

$$F(z) = z^{-1/2}G(-i\log z) = \frac{1}{\sqrt{2\pi}}\, z^{-1/2}\int_{-\infty}^{+\infty} e^{-it\log z}\mathfrak{F}(t)\,dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} z^{-1/2-it}\mathfrak{F}(t)\,dt.$$

Let $\mathfrak{F}(t)$ and $F(z)$ be related as above, and define $G(z)$ by (8), so $G(z) = e^{iz/2}F(e^{iz})$. Define

$$(9)\qquad \begin{aligned} G(0+iy) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{t(0+iy)}\mathfrak{F}(t)\,dt, \\ G(\pi+iy) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{t(\pi+iy)}\mathfrak{F}(t)\,dt \end{aligned}$$

where the integrals are taken in the mean square sense. By Parseval's formula, $\lim G(x+iy) = G(0+iy)$ as $x \searrow 0$ and $\lim G(x+iy) = G(\pi+iy)$ as $x \nearrow \pi$ in the metric of $L^2(-\infty, \infty)$. The limits hold a.e. also because $F(x) = \lim F(x+iy)$ as $y \searrow 0$ nontangentially a.e. Therefore the relation

$$G(\theta+iy) = e^{i(\theta+iy)/2}F(e^{i(\theta+iy)})$$

holds not only for $0 < \theta < \pi$, but also a.e. when $\theta = 0, \pi$. Now if $0 \le \theta \le \pi$, then

$$\int_{-\infty}^{+\infty} |G(\theta+iy)|^2\,dy = \int_{-\infty}^{+\infty} e^{-y}|F(e^{-y}e^{i\theta})|^2\,dy = \int_0^\infty |F(re^{i\theta})|^2\,dr.$$

But by Parseval's formula

$$\int_{-\infty}^{+\infty} |G(\theta+iy)|^2\,dy = \int_{-\infty}^{+\infty} e^{2\theta t}|\mathfrak{F}(t)|^2\,dt.$$

so (3)–(5) follow. Using (8) when $0 < \theta < \pi$ and (9) when $\theta = 0, \pi$, we obtain a.e.

$$\mathcal{F}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ity} e^{-\theta t} G(\theta + iy)\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ity} e^{i\theta(\frac{1}{2}+it)} e^{-\frac{1}{2}y} F(e^{-y}e^{i\theta})\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} (re^{i\theta})^{-\frac{1}{2}+it} F(re^{i\theta}) e^{i\theta}\, dr,$$

where the integrals are taken in the mean square sense. This yields (6) and completes the proof of (i). The assertions in (ii) follow directly from (i).

3. **Generalization.** Let Δ be a Borel subset of the real line such that neither Δ nor Δ^c is a Lebesgue null set. In the terminology of [5], a Cayley inner function mapping Δ on $(0, \infty)$ is any function $\xi(z)$ which is analytic and satisfies $\xi(z^*) = \xi(z)^*$ for $z \neq z^*$ and

(i) $\operatorname{Im} \xi(z) > 0$ for $y > 0$,

(ii) $\xi(x + i0) = \xi(x - i0)$ a.e. on $(-\infty, \infty)$, and

(iii) $\xi(x) \overset{\text{def}}{=} \xi(x + i0) = \xi(x - i0)$ satisfies $\xi(x) > 0$ a.e. on Δ and $\xi(x) < 0$ a.e. on Δ^c.

The general form of such a function [5, Theorem 2.2] is given by

$$\xi(z) = -1\Big/\exp\left(k + \int_\Delta \frac{1 + tz}{t - z} \frac{dt}{1 + t^2}\right)$$

where k is a real number. We understand that some such function is chosen and held fixed in the discussion. When $\Delta = \bigcup_1^r (a_j, b_j)$ where $-\infty < a_1 < b_1 < a_2 < b_2 < \cdots < a_r < b_r < \infty$, a convenient choice is

$$\xi(z) = -1\Big/\exp\left(\int_\Delta \frac{dt}{t - z}\right) = -\prod_1^r \frac{a_j - z}{b_j - z}.$$

If $\Delta = (0, \infty)$, then necessarily $\xi(z) = rz$ where $0 < r < \infty$, and we may choose $\xi(z) = z$.

We introduce the notation

$$l(\alpha, \beta) = \frac{\xi(\alpha) - \xi(\beta)^*}{\alpha - \beta^*} \quad \text{and} \quad l(t, \beta) = \frac{\xi(t) - \xi(\beta)^*}{t - \beta^*}$$

for $\alpha \neq \alpha^*$, $\beta \neq \beta^*$, and t real. As noted in [5], "composition with $\xi(t)$" is a meaningful operation in the class of a.e. defined functions on the real line. By [5, Theorem 3.3], if $f(t) \in L^1(-\infty, \infty)$ and $g(t) \in L^1(0, \infty)$, and $\alpha \neq \alpha^*$, $\beta \neq \beta^*$, then

$$\int_{-\infty}^{+\infty} l(t, \alpha) l(t, \beta)^* f(\xi(t))\, dt = l(\beta, \alpha) \int_{-\infty}^{+\infty} f(t)\, dt, \tag{10}$$

and

$$\int_\Delta l(t,\alpha)l(t,\beta)^* g(\xi(t))\,dt = l(\beta,\alpha)\int_0^\infty g(t)\,dt, \tag{11}$$

where the integrals on the left are absolutely convergent.

Theorem 2. (i) *If* $\mathfrak{F}(t)$ *is a measurable function on* $(-\infty, \infty)$ *such that*

$$\int_{-\infty}^{+\infty} (1 + e^{2\pi\xi(t)})|\mathfrak{F}(t)|^2\,dt < \infty, \tag{12}$$

then

$$F(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \frac{\xi(t)-\xi(z)}{t-z}\,\xi(z)^{-\frac{1}{2}-i\xi(t)}\mathfrak{F}(t)\,dt \tag{13}$$

defines a function in H^2 *whose boundary function* $F(x) = F(x+i0)$ *satisfies*

$$\int_\Delta |F(x)|^2\,dx = \int_{-\infty}^{+\infty} |\mathfrak{F}(t)|^2\,dt \tag{14}$$

and

$$\int_{\Delta^c} |F(x)|^2\,dx = \int_{-\infty}^{+\infty} e^{2\pi\xi(t)}|\mathfrak{F}(t)|^2\,dt. \tag{15}$$

Conversely, if $F(z)$ *is an* H^2 *function, there exists an essentially unique function* $\mathfrak{F}(t)$ *satisfying* (12) *such that* (13)–(15) *hold. The inversion formula*

$$\mathfrak{F}(t) = \frac{1}{\sqrt{2\pi}}\left(\lim_{\epsilon\searrow 0}\int_{\Delta_+} + \lim_{\epsilon\nearrow 0}\int_{\Delta_-}\right)\frac{\xi(x)-\xi(t+i\epsilon)}{x-t-i\epsilon}\,\xi(x)^{-\frac{1}{2}+i\xi(t+i\epsilon)}F(x)\,dx \tag{16}$$

holds a.e. and in the metric of $L^2(-\infty,\infty)$, *where* $\Delta_+ = \{x\colon \xi(x) > 1\}$ *and* $\Delta_- = \{x\colon 0 < \xi(x) < 1\}$.

(ii) *For every* $F(x) \in L^2(\Delta)$, (16) *defines a function* $\mathfrak{F}(t) \in L^2(-\infty,\infty)$. *There exists a function* $F(z)$ *in* H^2 *such that* $F(x+i0) = F(x)$ *a.e. on* Δ *if and only if* $\mathfrak{F}(t)$ *satisfies* (12), *and then* (13)–(15) *hold.*

Proof. Let $W(t) = 1 + e^{2\pi\xi(t)}$ and let $L^2(W)$ denote the Lebesgue space associated with the measure $W\,dx$ on $(-\infty,\infty)$. We first exhibit a special dense subspace of $L^2(W)$. Define $L^2(W_0)$ similarly for $W_0(t) = 1 + e^{2\pi t}$. We assert that functions of the form

$$\mathfrak{F}(t) = l(t,w)\mathfrak{F}_0(\xi(t)), \tag{17}$$

where $w \neq w^*$ and $\mathfrak{F}_0 \in L^2(W_0)$, belong to $L^2(W)$ and span a dense subspace of $L^2(W)$. By (10) such functions belong to $L^2(W)$. To see that there are enough to span a dense subspace, choose $\mathcal{F}_0(t) = \chi_{(-A,A)}(t)/(t - \xi(w)^*)$, where $w \neq w^*$ and $A > 0$. It follows that $\chi_{\{x:\ |\xi(x)|<A\}}(t)/(t - w^*)$ belongs to the set, and so density follows by routine arguments.

Next we exhibit a special dense subspace of H^2. Namely, we assert that functions of the form

$$F(z) = l(z, w)F_0(\xi(z)), \qquad y > 0, \tag{18}$$

where $w \neq w^*$ and $F_0 \in H^2$, belong to H^2 and span a dense subspace of H^2. Assume $F_0 \in H^2$, $w \neq w^*$, and define F by (18). By Cauchy's formula,

$$F_0(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{F_0(t)}{t - z}\,dt, \qquad y > 0.$$

By (10) the function $F(x) = l(x, w)F_0(\xi(x))$ is in $L^2(-\infty, \infty)$. We note that by a theorem of Ryff [6], $F(x) = F(x + i0)$ is the boundary function of $F(z)$. This justifies the notation, but logically it is not needed here. It will be used later. We obtain

$$F(z) = \int_{-\infty}^{+\infty}\frac{F(z)}{t - z}\,dt, \qquad y > 0,$$

from the Cauchy representation of $F_0(z)$ using (10). Thus $F \in H^2$. Choosing $F_0(z) = 1/[z - \xi(w)^*]$, we obtain $F(z) = 1/(z - w^*)$, and so density follows.

Next we show that (13) defines an isometry $U_+\colon \mathcal{F}(t) \to F(z)$ mapping $L^2(W)$ onto H^2. Straightforward estimates show that for each fixed z in the upper half-plane the integral in (13) is a continuous linear functional of $\mathcal{F}$ on $L^2(W)$. By what was proved above, it therefore suffices to check that

$$\begin{aligned}&\langle l(t, \alpha)\mathcal{F}_0(\xi(t)), l(t, \beta)\mathcal{G}_0(\xi(t))\rangle_{L^2(W)}\\ &\qquad = \langle l(z, \alpha)F_0(\xi(z)), l(z, \beta)G_0(\xi(z))\rangle_{H^2}\end{aligned} \tag{19}$$

and

$$\begin{aligned}&l(z, w)F_0(\xi(z))\\ &\quad = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} l(t, z)^*\xi(z)^{-\frac{1}{2}-i\xi(t)}l(t, w)\mathcal{F}_0(\xi(t))\,dt, \qquad y > 0,\end{aligned} \tag{20}$$

for any nonreal numbers α, β, w and any $\mathcal{F}_0$, $\mathcal{G}_0 \in L^2(W_0)$ and F_0, $G_0 \in H^2$ which are connected by

$$F_0(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} z^{-\frac{1}{2}-it}\mathcal{F}_0(t)\,dt, \qquad G_0(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} z^{-\frac{1}{2}-it}\mathcal{G}_0(t)\,dt \tag{21}$$

for $y > 0$. Here, of course, implicit use is made of Theorem 1 in knowing that F_0 and G_0 exist given $\mathfrak{F}_0$ and $\mathcal{G}_0$, and conversely any F_0 and G_0 arise from some $\mathfrak{F}_0$ and $\mathcal{G}_0$. In fact, Theorem 1 combined with (10) yields (19). We obtain (20) from (10) and (21). The details are left to the reader.

Let $\mathcal{C}$ be the Hilbert space with reproducing kernel $l(z, w)$, $w \neq w^*$, $z \neq z^*$. The elements of $\mathcal{C}$ are functions separately analytic for $y > 0$ and $y < 0$. We see from (10) and (11) that the linear transformations

$$P\colon \mathcal{C} \otimes L^2(0, \infty) \to L^2(\Delta) \quad \text{and} \quad Q\colon \mathcal{C} \otimes L^2(-\infty, \infty) \to L^2(-\infty, \infty)$$

specified by

$$P\colon l(z, w) \otimes f(t) \to l(t, w) f(\xi(t)) \quad \text{and} \quad Q\colon l(z, w) \otimes g(t) \to l(t, w) g(\xi(t)),$$

where w is any nonreal number, $f \in L^2(0, \infty)$, $g \in L^2(-\infty, \infty)$, and $\otimes$ denotes the Hilbert space tensor product, are isometric isomorphisms. Next we place $\mathcal{C} \otimes L^2(0, \infty)$ and $\mathcal{C} \otimes L^2(-\infty, \infty)$ in linear isometric correspondence by $I \otimes M$, where M is the Mellin transform, so

$$I \otimes M\colon l(z, w) \otimes f(t) \to l(z, w) \otimes \underset{T\to\infty}{\text{l.i.m.}} \frac{1}{\sqrt{2\pi}} \int_{1/T}^{T} x^{-\frac{1}{2}+it} f(x)\, dx$$

and

$$(I \otimes M)^{-1}\colon l(z, w) \otimes g(t) \to l(z, w) \otimes \underset{T\to\infty}{\text{l.i.m.}} \frac{1}{\sqrt{2\pi}} \int_{-T}^{+T} t^{-\frac{1}{2}-ix} g(x)\, dx$$

for each $f \in L^2(0, \infty)$ and $g \in L^2(-\infty, \infty)$.

By construction $U = P(I \otimes M)^{-1} Q^{-1}$ maps $L^2(-\infty, \infty)$ isometrically onto $L^2(\Delta)$. Note that $L^2(W)$ is contained in $L^2(-\infty, \infty)$ and the inclusion mapping is bounded by 1. We assert that if $\mathfrak{F} \in L^2(W)$, $U_+\colon \mathfrak{F}(t) \to F(z)$, and $U\colon \mathfrak{F}(t) \to F_\Delta(x)$, then $F_\Delta(x)$ is the restriction to Δ of the boundary function $F(x) = F(x + i0)$ of $F(z)$. It is enough to check this for a set which is dense in $L^2(W)$. It is true for functions of the form (17) by direct calculation. The general case follows by linearity and continuity. We now have the diagram

$$\begin{array}{ccc}
\mathcal{C} \otimes L^2(-\infty, \infty) & \xleftarrow{I\otimes M} & \mathcal{C} \otimes L^2(0, \infty) \\
\downarrow Q & & \downarrow P \\
L^2(-\infty, \infty) & \xrightarrow{U} & L^2(\Delta) \\
\text{injection}\ \uparrow & & \uparrow\ \text{restriction to } \Delta \\
L^2(W) & \xrightarrow{U_+} & H^2
\end{array}$$

where $I \otimes M$, P, Q, U, and U_+ are isometric isomorphisms.

Now if $\mathfrak{F} \in L^2(W)$ and if F in H^2 is the corresponding function given by (13), then because U_+ and U are unitary

$$\int_{-\infty}^{+\infty} |F(t)|^2\, dt = \int_{-\infty}^{+\infty} [1 + e^{2\pi\xi(t)}]|\mathfrak{F}(t)|^2\, dt$$

and

$$\int_{\Delta} |F(t)|^2\, dt = \int_{-\infty}^{+\infty} |\mathfrak{F}(t)|^2\, dt,$$

so (14) and (15) hold.

The inversion formula (16) involves an explicit calculation of U^{-1}, which we give next. If $\mathfrak{F} \in L^2(-\infty, \infty)$, then

$$\mathfrak{F}(t) = \lim_{\epsilon \searrow 0} \frac{\epsilon}{\pi} \int_{-\infty}^{+\infty} \frac{\mathfrak{F}(x)}{(x-t)^2 + \epsilon^2}\, dx$$

a.e. and in the metric of $L^2(-\infty, \infty)$. We assert that if $U\mathfrak{F} = F$, then

$$\begin{aligned} \frac{\epsilon}{\pi} \int_{-\infty}^{+\infty} \frac{\mathfrak{F}(x)}{(x-t)^2 + \epsilon^2}\, dx &= \frac{1}{\sqrt{2\pi}} \int_{\Delta_+} l(x, t + i\epsilon)^* \xi(x)^{-\frac{1}{2} + i\xi(t+i\epsilon)} F(x)\, dx \\ &\quad + \frac{1}{\sqrt{2\pi}} \int_{\Delta_-} l(x, t - i\epsilon)^* \xi(x)^{-\frac{1}{2} + i\xi(t-i\epsilon)} F(x)\, dx \end{aligned} \tag{22}$$

for t real, $\epsilon > 0$. Now for fixed t and ϵ, the left side of (22) equals

$$\begin{aligned} &\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \mathfrak{F}(x)[(x - t - i\epsilon)^{-1} - (x - t + i\epsilon)^{-1}]\, dx \\ &\qquad = \frac{1}{2\pi i} \langle U^{-1}F, (x - t + i\epsilon)^{-1} - (x - t - i\epsilon)^{-1} \rangle_{L^2(-\infty,\infty)} \\ &\qquad = \frac{1}{2\pi i} \langle F, U(x - t + i\epsilon)^{-1} - U(x - t - i\epsilon)^{-1} \rangle_{L^2(\Delta)}, \end{aligned}$$

where here and in the rest of the proof x is used as a dummy variable. We can derive (22) and consequently (16) by calculating $U(x - t + i\epsilon)^{-1}$ and $U(x - t - i\epsilon)^{-1}$. We find that

$$Q^{-1}\colon (x - t + i\epsilon)^{-1} \longrightarrow l(z, t + i\epsilon) \otimes [x - \xi(t + i\epsilon)^*]^{-1},$$

$$(I \otimes M)^{-1} Q^{-1}\colon (x - t + i\epsilon)^{-1} \longrightarrow -i\sqrt{2\pi}\, l(z, t + i\epsilon) \otimes x^{-\frac{1}{2} - i\xi(t+i\epsilon)^*} \chi_{(1,\infty)}(x),$$

and

$$U\colon (x - t + i\epsilon)^{-1} \longrightarrow \begin{cases} -i\sqrt{2\pi}\, l(x, t + i\epsilon)\xi(x)^{-\frac{1}{2} - i\xi(t+i\epsilon)^*} & \text{on } \Delta_+, \\ 0 \quad \text{on } \Delta_-. \end{cases}$$

Similarly

$$U\colon (x - t - i\epsilon)^{-1} \longrightarrow \begin{cases} 0 \quad \text{on } \Delta_+, \\ i\sqrt{2\pi}\, l(x, t - i\epsilon)\xi(x)^{-\frac{1}{2} - i\xi(t-i\epsilon)^*} & \text{on } \Delta_-. \end{cases}$$

This implies (22) and completes the proof of (i). The assertions in (ii) are evident from our constructions.

REFERENCES

1. G. M. Golusin and W. J. Krylov, *Generalization of a formula of Carleman and its application to analytic continuation*, Mat. Sb. 40 (1933), 144–149.

2. M. G. Kreĭn and P. Ja. Nudel'man, *Certain new problems for functions of a Hardy class and for continual families of functions with double orthogonality*, Dokl. Akad. Nauk SSSR 209 (1973), 537–540 = Soviet Math. Dokl. 14 (1973), 435–439. MR 47 #9156.

3. D. J. Patil, *Representation of H^p-functions*, Bull. Amer. Math. Soc. 78 (1972), 617–620. MR 45 #7069.

4. M. Rosenblum and J. Rovnyak, *Restrictions of analytic functions*. I, Proc. Amer. Math. Soc. 48 (1975), 113–119.

5. ———, *Cayley inner functions and best approximation*, J. Approximation Theory (to appear).

6. J. V. Ryff, *Subordinate H^p functions*, Duke Math. J. 33 (1966), 347–354. MR 33 #289.

7. A. Steiner, *Zum Mechanismus der Quasianalytizität gewisser Randfunktionen auf endlichen Intervallen*, Ann. Acad. Sci Fenn. Ser. A I No. 459 (1970). MR 41 #8681.

8. E. C. Titchmarsh, *Theory of Fourier integrals*, 2nd ed., Oxford Univ. Press, New York, 1948.

9. C. van Winter, *Fredholm equations on a Hilbert space of analytic functions*, Trans. Amer. Math. Soc. 162 (1971), 103–139.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF VIRGINIA, CHARLOTTESVILLE, VIRGINIA 22903