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NONFACTORIZATION OF FUNCTIONS IN BANACH SUBSPACES OF $L^1(G)$

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ABSTRACT. In this note we first prove a theorem on factorization of functions in certain subsets of $L^1(G)$, where G is a nondiscrete locally compact Abelian group with dual group $\hat{G}$. One of the corollaries of this theorem answers a question of R. Larsen concerning the algebras of functions with Fourier transforms in $L^p(\hat{G})$. The other corollaries contain nonfactorization results which sharpen some known theorems.

Throughout this note G denotes a nondiscrete locally compact Abelian group with dual group $\hat{G}$. For $A, B \subset L^1(G)$, the sets $\{f * g: f \in A, g \in B\}$ and $\{\sum_{i=1}^n f_i * g_i: f_i \in A, g_i \in B, n = 1, 2, 3, \cdots\}$ will be denoted by $A * B$ and $[A * B]$, respectively. For $1 \le p < \infty$, define $A^p(G) = \{f \in L^1(G): \hat{f} \in L^p(\hat{G})\}$ and $\|f\|_{A^p} = \|f\|_1 + \|\hat{f}\|_p$ for $f \in A^p(G)$. The Banach algebras $(A^p(G), \|\cdot\|_{A^p})$ have been studied by many authors (see the survey article by Larsen [2]). For $1 \le p < q < \infty$, we have $A^q(G) * A^p(G) \subset A^p(G)$, and Larsen [2] has raised the question: When, if ever, is $A^q(G) * A^p(G) = A^p(G)$? The answer to this question is given in a corollary (see Corollary 1 below) of the following theorem, the proof of which is based on an idea first used in Martin and Yap [3, p. 218].

Theorem 1. *Let A, B be subsets of $L^1(G)$ such that $\hat{A} \equiv \{\hat{f}: f \in A\} \subset L^p(\hat{G})$ for some $p \in (0, \infty)$ and $B \subset [A * B]$. Then $\hat{B} \subset L^r(\hat{G})$ for all $r \in (0, \infty]$.*

Proof. It is clear that $\hat{B} \subset L^r(\hat{G})$ for all $r \in [p, \infty]$. Now consider $h \in B$ and $r \in (0, p)$. Choose a positive integer N such that $p/2^N \le r$. By repeated use of the condition $B \subset [A * B]$, we can write $h = h_1 + \cdots + h_m$ with $h_i = f_1 * f_2 * \cdots * f_{2N} * g$, where $f_i \in A$, $g \in B$. Since $\hat{f}_i \in L^p(\hat{G})$, it follows from Hölder's inequality that

$$(f_1 * f_2 * \cdots * f_{2N})\hat{} = \hat{f}_1\hat{f}_2 \cdots \hat{f}_{2N}$$

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is in $L^{p/2^N}(\hat{G})$. Since $\hat{g}$ is bounded, we have $\hat{h}_i \in L^{p/2^N}(\hat{G})$ and hence $\hat{h}_i \in L^r(\hat{G})$. Thus $\hat{h} \in L^r(\hat{G})$ for all $r \in (0, p)$. This completes the proof.

We now give some sample corollaries of Theorem 1.

Corollary 1. *Let $1 \le p < q < \infty$. Then $[A^q(G) * A^p(G)]$ is a proper subspace of $A^p(G)$.*

Proof. Since $A^p(G)$ is a "character" Segal algebra, it follows that $A^p(G)\hat{} \not\subset L^r(\hat{G})$ for some r in $(0, \infty)$ (see Wang [4] for details). The conclusion, in view of Theorem 1, is now clear.

For $1 < p \le \infty$, define $B^p(G) = L^1(G) \cap L^p(G)$ and $\|f\|_{B^p} = \|f\|_1 + \|f\|_p$ for $f \in B^p(G)$. It is well known [5], [4], [1] that $[B^p(G) * B^p(G)]$ is a *proper* subspace of $B^p(G)$ for $1 < p < \infty$ (and G nondiscrete). Now we can prove the following stronger result.

Corollary 2. *For $1 < p < q \le \infty$, $[B^p(G) * B^q(G)]$ is a proper subspace of $B^q(G)$.*

Proof. Clearly $[B^p(G) * B^q(G)] \subset B^q(G)$. By the Hausdorff-Young theorem we have $B^p(G)\hat{} \subset L^r(\hat{G})$ for some r. The desired conclusion follows immediately from Theorem 1 and the fact that $B^q(G)$ is a "character" Segal algebra (see Wang [4]).

Let T be the circle group. For each positive integer k, define $C^k(T)$ to be the space of all functions with k continuous derivatives, and norm $C^k(T)$ by

$$\|f\|_{C^k} = \max_{0 \le j \le k} \max_{x \in T} |f^{(j)}(x)|.$$

It is easy to see that we have the chain [4, p. 234]

$$(1) \quad \cdots \subset C^{k+1}(T) \subset C^k(T) \subset \cdots \subset C(T) \subset \cdots \subset L^r(T) \subset L^s(T) \subset \cdots,$$

where $r > s > 1$. Even though $C^k(T)$, $k \ge 1$, is not a "character" Segal algebra, we nevertheless have $C^k(T)\hat{} \not\subset L^r(\hat{T})$ for some r in $(0, \infty)$ (see [4]). The following corollary, in view of Theorem 1, is now immediate.

Corollary 3. *If A, B are any two sets from the chain* (1) *with $A \supset B$, then $[A * B]$ is a proper subspace of B.*

Remark. The special case $A = B$ of Corollary 3 was obtained by Wang [4]. It is now clear that all the nonfactorization results in Wang's paper can be extended in the same way.

From Theorem 1 we see that if A, B are subsets of $L^1(G)$ with

$\hat{A} \subset L^p(\hat{G})$, $\hat{B} \not\subset L^r(\hat{G})$ for some p, r in $(0, \infty)$, then $B \not\subset [A * B]$. Thus, for the purpose of obtaining nonfactorization results, it is of interest to have conditions on B which would imply $\hat{B} \not\subset L^r(\hat{G})$ for some r in $(0, \infty)$. The condition (called Property P) given by Wang [4, p. 235] for L^1-dense Banach *subalgebras* $(B, \|\cdot\|_B)$ of $L^1(G)$ is also meaningful when $(B, \|\cdot\|_B)$ is a Banach subspace of $L^1(G)$. The following extension of the nonfactorization theorem in Wang [4, Theorem 4.1] can be proved by using Theorem 1 above and the idea used in Wang's proof.

Theorem 2. *Let A be a subset of $L^1(G)$ with $\hat{A} \subset L^p(\hat{G})$ for some p in $(0, \infty)$. Let B be a subspace of $L^1(G)$ such that $(B, \|\cdot\|_B)$ is Banach space having Property* P *and $\|\cdot\|_1 \leq M\|\cdot\|_B$ for some constant M. Then $B \not\subset [A * B]$.*

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REFERENCES

1. J. T. Burnham, *Nonfactorization in subsets of the measure algebra*, Proc. Amer. Math. Soc. 35 (1972), 104–106. MR 45 #7394.

2. R. Larsen, *The algebras of functions with Fourier transforms in L_p: A survey*, University of Oslo, 1973, (preprint).

3. J. C. Martin and L. Y. H. Yap, *The algebra of functions with Fourier transforms in L^p*, Proc. Amer. Math. Soc. 24 (1970), 217–219. MR 40 #646.

4. H. C. Wang, *Nonfactorization in group algebras*, Studia Math. 42 (1972), 231–241. MR 46 #2355.

5. L. Y. H. Yap, *Ideals in subalgebras of the group algebras*, Studia Math. 35 (1970), 165–175. MR 42 #4968.

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