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INDEPENDENCE AND ADDITIVE ENTROPY

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ABSTRACT. The relationship between additive entropy and independence is worked out for ergodic transformations on a Lebesgue space. Examples are given on the behavior of the deterministic part of an ergodic transformation.

In this note we suppose that T is an ergodic automorphism of a Lebesgue space (Ω, F, μ). We let $\alpha, \beta, \gamma, \ldots$ denote finite or countable partitions of X of finite entropy, i.e. $-\Sigma\{\mu(A)\log\mu(A)|A \in \alpha\} < \infty$. We let $\alpha \vee \beta$ denote the common refinement of α and β, $\alpha^- = \bigvee_{i=1}^{\infty} T^{-i}\alpha$, $\alpha_T = \bigvee_{i=-\infty}^{\infty} T^i\alpha$ and $\alpha' = \bigwedge_{i>0} T^{-i}\alpha^-$. Of course α^-, α_T, and α' are generally uncountable partitions. We let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots, \mathcal{A}^-, \mathcal{A}_T$, etc. be the σ-algebras corresponding to $\alpha, \beta, \gamma, \ldots, \alpha^-, \alpha_T$ etc. See [4] for the details of this correspondence, as well as the justification of the basic entropy computations which we make. We let $\mathcal{H}(\alpha|\beta)$ be the conditional entropy of α given β; we let $h(\alpha) = h(\alpha, T) = \mathcal{H}(\alpha|\alpha^-)$, and we let ϵ and ν be the partition into singletons and the partition $\{\Omega\}$ respectively.

It is well known that partitions α and β of finite entropy are independent if and only if $\mathcal{H}(\alpha \vee \beta) = \mathcal{H}(\alpha) + \mathcal{H}(\beta)$. One method of proof is to write the conditional entropy formula $\mathcal{H}(\alpha \vee \beta) = \mathcal{H}(\alpha|\beta) + \mathcal{H}(\beta)$, and then to show that $\mathcal{H}(\alpha|\beta) = \mathcal{H}(\alpha)$ if α and β are independent (using the strict convexity of $\eta(x) = -x \log x$). In [7, Lemma 3], Rohlin and Sinai established an analogue of the conditional entropy formula for transformations which asserts $\mathcal{H}(\alpha \vee \beta|\alpha^- \vee \beta^-) = \mathcal{H}(\alpha|\alpha^- \vee \beta_T) + \mathcal{H}(\beta|\beta^-)$. In [1, Lemma 2.3] we showed that this formula could be used to show that the partitions α_T and β_T were independent (in [1] we referred to the corresponding algebras) providing $h(\alpha \vee \beta) = h(\alpha) + h(\beta)$ and α (or β) generates a system of completely positive entropy i.e. $\alpha' = \nu$. This result includes an earlier result of Pinsker [6, Theorem 5] which asserts that α_T and β_T are independent

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if $\alpha' = \nu$ and $h(\beta) = 0$. In this note we will show that α_T and β_T are independent if and only if $h(\alpha \vee \beta) = h(\alpha) + h(\beta)$ and α' and β' are independent. We will make some observations concerning the mapping $\alpha \to \alpha'$.

We will need the following lemma; the proof appears in [1, Lemma 2.2].

Lemma. *If α and β are partitions of finite entropy and $h(\alpha \vee \beta) = h(\alpha) + h(\beta)$ then $\mathcal{H}(T^k\beta_n|\alpha^-) = \mathcal{H}(T^k\beta_n|\alpha')$ for every k and every nonnegative n, where $\beta_n = \beta \vee \ldots \vee T^n\beta$.*

Theorem 1. *If α and β are partitions of finite entropy then α_T and β_T are independent if and only if α' and β' are independent and $h(\alpha \vee \beta) = h(\alpha) + h(\beta)$.*

Proof. It is well known that the independence of α_T and β_T implies the independence of all coarser partitions (such as α' and β') and the addition of entropy, so we must only establish the converse.

We first assume $\beta' = \beta^-$ i.e. $h(\beta) = 0$. This implies $T^k\beta_n \leq \beta'$ and, since α' and β' are independent, $\mathcal{H}(T^k\beta_n) = \mathcal{H}(T^k\beta_n|\alpha') = \mathcal{H}(T^k\beta_n|\alpha^-)$ for all k and n. Then

$$\mathcal{H}(T^k\beta_n|T^j\alpha^-) = \mathcal{H}(T^{k-j}\beta_n|\alpha^-) = \mathcal{H}(T^{k-j}\beta_n) = \mathcal{H}(T^k\beta_n).$$

Let $j \to +\infty$ and we find $\mathcal{H}(T^k\beta_n|\alpha_T) = \mathcal{H}(T^k\beta_n)$, and we conclude $T^k\beta_n$ is independent of α_T for each k and n, which implies β_T is independent of α_T.

We now drop the assumption that $h(\beta) = 0$. From the Lemma, $\mathcal{H}(T^k\alpha_n|\beta^-) = \mathcal{H}(T^k\alpha_n|\beta')$. We let γ be a partition of finite entropy such that $\gamma_T = \beta'$ (such a partition exists by a theorem of Rohlin; see [4, Theorem 7.4], also [3]). Then $\gamma' = \gamma_T = \beta'$ is independent of α', and $h(\gamma) = 0$, so we conclude (from the previous paragraph) that α_T and $\gamma_T = \beta'$ are independent. In particular, $\mathcal{H}(T^k\alpha_n|\beta') = \mathcal{H}(T^k\alpha_n)$. We use the Lemma to conclude $\mathcal{H}(T^k\alpha_n|\beta^-) = \mathcal{H}(T^k\alpha_n)$, and a repetition of the calculations and reasoning of the previous paragraph shows that α_T and β_T are independent. Q.E.D.

Corollary. *Two ergodic processes are disjoint (see [2]) if and only if at least one of the two processes has zero entropy and their Pinsker (maximal zero entropy) factors are disjoint.*

Proof. In [2], Furstenberg showed two processes of positive entropy cannot be disjoint. Moreover, it suffices to show that the corollary is correct if the processes have finite entropy, since an arbitrary ergodic process is the inverse limit of processes of finite entropy. Further, the formula $\mathcal{H}(\alpha \vee \beta|\alpha^- \vee \beta^-) = \mathcal{H}(\alpha|\alpha^- \vee \beta_T) + \mathcal{H}(\beta|\beta^-)$ can be used to show that entropy is always additive if one of the two processes has zero entropy. The result now follows from the theorem and the fact that disjointness is preserved under passage to factor processes. Q.E.D.

Because questions of independence of α_T and β_T can sometimes be reduced to questions of independence of α' and β', it is useful to examine the mapping $\alpha \to \alpha'$. We provide a theorem and an example, which may be folklore, but which we did not find in print.

Theorem 2. *If α_T and β_T are independent then* $(\alpha \vee \beta)' = \alpha' \vee \beta'$ (mod 0).

Proof. We will work with the corresponding algebras. We must show that $(\mathfrak{A} \vee \mathfrak{B})' \leq \mathfrak{A}' \vee \mathfrak{B}'$, as the reverse inclusion is obvious. It suffices to show that if $\mathfrak{X}$ is a finite subalgebra of $\mathfrak{A}_T \vee \mathfrak{B}_T$ such that $\mathcal{H}(\mathfrak{X}|\mathfrak{X}^-) = 0$ then $\mathfrak{X} \leq \mathfrak{A}' \vee \mathfrak{B}'$ or, equivalently, $\mathcal{H}(\mathfrak{X}|\mathfrak{A}' \vee \mathfrak{B}') = 0$. Suppose that $\mathfrak{X}$ is such a finite subalgebra, and suppose $\mathfrak{A}_0$ is a finite subalgebra of $\mathfrak{A}'$. Since α_T and β_T are independent, we know $\mathcal{H}(\mathfrak{A}_0|\mathfrak{B}') = \mathcal{H}(\mathfrak{A}_0|\mathfrak{B}^-)$ (they both equal $\mathcal{H}(\mathfrak{A}_0)$). Since $\mathcal{H}(\mathfrak{X} \vee \mathfrak{A}_0|(\mathfrak{X} \vee \mathfrak{A}_0)^-) = 0$, we know

$$h((\mathfrak{X} \vee \mathfrak{A}_0) \vee \mathfrak{B}, T) = h(\mathfrak{X} \vee \mathfrak{A}_0, T) + h(\mathfrak{B}, T)$$

and so the Lemma for Theorem 1 implies $\mathcal{H}(\mathfrak{X} \vee \mathfrak{A}_0|\mathfrak{B}^-) = \mathcal{H}(\mathfrak{X} \vee \mathfrak{A}_0|\mathfrak{B}')$. Thus

$$\begin{aligned}\mathcal{H}(\mathfrak{X}|\mathfrak{A}_0 \vee \mathfrak{B}') &= \mathcal{H}(\mathfrak{X} \vee \mathfrak{A}_0|\mathfrak{B}') - \mathcal{H}(\mathfrak{A}_0|\mathfrak{B}') \\ &= \mathcal{H}(\mathfrak{X} \vee \mathfrak{A}_0|\mathfrak{B}^-) - \mathcal{H}(\mathfrak{A}_0|\mathfrak{B}^-) = \mathcal{H}(\mathfrak{X}|\mathfrak{A}_0 \vee \mathfrak{B}^-).\end{aligned}$$

Thus $\mathcal{H}(\mathfrak{X}|\mathfrak{A}_0 \vee \mathfrak{B}') = \mathcal{H}(\mathfrak{X}|\mathfrak{A}_0 \vee \mathfrak{B}^-)$ for an arbitrary finite subalgebra $\mathfrak{A}_0 \leq \mathfrak{A}'$, which implies $\mathcal{H}(\mathfrak{X}|\mathfrak{A}' \vee \mathfrak{B}') = \mathcal{H}(\mathfrak{X}|\mathfrak{A}' \vee \mathfrak{B}^-)$. The same argument applies with $\mathfrak{B}$ replaced by $T\mathfrak{B}$ (and $(T\mathfrak{B})' = \mathfrak{B}'$) so

$$\mathcal{H}(\mathfrak{X}|\mathfrak{A}' \vee \mathfrak{B}') = \mathcal{H}(\mathfrak{X}|\mathfrak{A}' \vee T^i\mathfrak{B}^-) \longrightarrow \mathcal{H}(\mathfrak{X}|\mathfrak{A}' \vee \mathfrak{B}_T)$$

as $i \to \infty$. Now let $\mathfrak{B}_0$ be a finite subalgebra of $\mathfrak{B}_T$, and observe $\mathcal{H}(\mathfrak{X}|\mathfrak{A}' \vee \mathfrak{B}_0) = \mathcal{H}(\mathfrak{X} \vee \mathfrak{B}_0|\mathfrak{A}') - \mathcal{H}(\mathfrak{B}_0|\mathfrak{A}')$. By independence $\mathcal{H}(\mathfrak{B}_0|\mathfrak{A}') = \mathcal{H}(\mathfrak{B}_0|\mathfrak{A}_T)$. By using independence of $\mathfrak{A}_T$ and $\mathfrak{B}_T$, and the fact that $(\mathfrak{X}|\mathfrak{X}^-) = 0$, we calculate

$$\mathcal{H}((\mathcal{X} \vee \mathcal{B}_0) \vee \mathcal{A} | (\mathcal{X} \vee \mathcal{B}_0)^- \vee \mathcal{A}^-) = \mathcal{H}(\mathcal{X} \vee (\mathcal{B}_0 \vee \mathcal{A}) | \mathcal{X}^- \vee (\mathcal{B}_0 \vee \mathcal{A})^-)$$
$$= \mathcal{H}(\mathcal{B}_0 \vee \mathcal{A} | (\mathcal{B}_0 \vee \mathcal{A})^-) = \mathcal{H}(\mathcal{B}_0 | \mathcal{B}_0^-) + \mathcal{H}(\mathcal{A} | \mathcal{A}^-)$$
$$= \mathcal{H}(\mathcal{X} \vee \mathcal{B}_0 | (\mathcal{X}_0 \vee \mathcal{B}_0)^-) + \mathcal{H}(\mathcal{A} | \mathcal{A}^-)$$

and so, by applying the Lemma to $\mathcal{X} \vee \mathcal{B}_0$ and $\mathcal{A}$, we see that $\mathcal{H}(\mathcal{X} \vee \mathcal{B}_0 | \mathcal{A}') = \mathcal{H}(\mathcal{X} \vee \mathcal{B}_0 | \mathcal{A}^-)$. Observing that the same argument applies to $T\mathcal{A}$, and taking limits, we see $\mathcal{H}(\mathcal{X} \vee \mathcal{B}_0 | \mathcal{A}') = \mathcal{H}(\mathcal{X} \vee \mathcal{B}_0 | \mathcal{A}_T)$. Thus, expanding, replacing, and collecting as before, we see that $\mathcal{H}(\mathcal{X} | \mathcal{A}' \vee \mathcal{B}_0) = \mathcal{H}(\mathcal{X} | \mathcal{A}_T \vee \mathcal{B}_0)$. Since this holds for an arbitrary finite subalgebra $\mathcal{B}_0$ of $\mathcal{B}_T$, we see that $\mathcal{H}(\mathcal{X} | \mathcal{A}' \vee \mathcal{B}_T) = \mathcal{H}(\mathcal{X} | \mathcal{A}_T \vee \mathcal{B}_T)$. Thus

$$\mathcal{H}(\mathcal{X} | \mathcal{A}' \vee \mathcal{B}') = \mathcal{H}(\mathcal{X} | \mathcal{A}' \vee \mathcal{B}_T) = \mathcal{H}(\mathcal{X} | \mathcal{A}_T \vee \mathcal{B}_T) = 0,$$

since $\mathcal{X} \leq \mathcal{A}_T \vee \mathcal{B}_T$. Q.E.D.

Corollary. *If* α *and* β *have finite entropy, if* $\alpha' = \beta' = \nu$, *and if* $h(\alpha \vee \beta) = h(\alpha) + h(\beta)$ *then* $(\alpha \vee \beta)' = \nu$.

Proof. This is clear from Theorems 1 and 2.

As reported by Parry in [5], Ken Thomas has shown that if α_T and β_T are not independent then $(\alpha \vee \beta)'$ may be different from $\alpha' \vee \beta'$. It is the purpose of the following example to show that in fact any zero entropy partition is possible for $(\alpha \vee \beta)'$ even though α_T and β_T define Bernoulli shifts.

Let $\Omega = \mathbf{Z}_2^{\mathbf{Z}}$; let T be the shift $(T\omega)_i = \omega_{i+1}$. Let μ be the measure for the fair coin Bernoulli shift, and let ν be a zero entropy T invariant measure (by [3], any zero entropy process can be realized in this manner). The phase space of our example is the product space $\Omega \times \Omega$. We let $A_i = \{(\omega, \omega'): \omega_0 = i\}$ $(i = 1, 2)$, $C_i = \{(\omega, \omega'): \omega'_0 = i\}$ $(i = 1, 2)$, $B_0 = \{(\omega, \omega'): \omega_0 = \omega'_0\}$, $B_1 = \{(\omega, \omega'): \omega_0 \neq \omega'_0\}$, $\alpha = \{A_1, A_2\}$, $\beta = \{B_1, B_2\}$, $\gamma = \{C_1, C_2\}$. Then $\alpha \vee \beta = \alpha \vee \gamma$, and $(\alpha \vee \gamma)' = \alpha' \vee \gamma' = \gamma' = \gamma_T$. Evidently $(\Omega \times \Omega, \mathcal{C}_T, T \times T, \mu \times \nu)$ is conjugate to (Ω, F, T, ν), and $(\Omega \times \Omega, \mathcal{A}_T, T \times T, \mu \times \nu)$ is Bernoulli. It remains to be seen that $(\Omega \times \Omega, \mathcal{B}_T, T \times T, \mu \times \nu)$ is Bernoulli. Let K be an atom of $\mathcal{B} \vee T\mathcal{B} \dots T^{n-1}\mathcal{B}$. There are integers $0 \leq \nu_1 < \nu_2 < \dots < \nu_l \leq n-1$ such that $(\omega, \omega') \in K$ if and only if $\omega_i = \omega'_i$ if i is among the ν, and $\omega_i \neq \omega'_i$ otherwise. But this implies that for each atom J of $\mathcal{A} \vee T\mathcal{A} \vee \dots \vee T^{n-1}\mathcal{A}$ there is exactly one atom J' of $\mathcal{C} \vee T\mathcal{C} \vee \dots \vee T^{n-1}\mathcal{C}$ such that $J \times J'$ intersects K, that $J_1 \neq J_2 \Rightarrow J'_1 \neq J'_2$, and that $K = \bigcup J \times J'$. But then

$$\mu \times \nu(K) = \sum (\mu \times \nu)(J \times J') = \sum \mu(J)\nu(J')$$

$$= \sum \left(\frac{1}{2}\right)^n \nu(J') = \left(\frac{1}{2}\right)^n \sum \nu(J') = \left(\frac{1}{2}\right)^n$$

This shows that $\mathcal{B}_T$ is Bernoulli, and completes the construction of the example.

We close with a question suggested by the example. Suppose X and Y are two processes, each of completely positive entropy, and suppose Z is a process of zero entropy such that $X \times Z$ is isomorphic to $Y \times Z$. Are X and Y isomorphic?

Added in proof. Recent work of J. P. Thouvenot implies that the answer to the above question is yes if X is Bernoulli. Recent work of Rudolph implies the answer is no in general, even if Z is finite.

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