LOGARITHMIC MEANS AND SUMMABILITY BY THE CIRCLE METHODS

M. R. PARAMESWARAN

ABSTRACT. It is proved that if the logarithmic means of a sequence $\{s_n\}$ satisfy a certain order condition then the sequence $\{s_n\}$ will be summable by every circle method (Kreisverfahren) stronger than convergence; the condition is shown to be a best possible one, for even summability by the collective circle method. A second set of conditions leading to the conclusion of summability by every circle method is also proved.

1. Introduction. Given a sequence $\{s_n\}$, its logarithmic means t_n ($n = 0, 1, \dots, \dots$) are defined by

$$(1) \quad t_0 = s_0, \quad t_1 = s_1 \quad \text{and} \quad t_n = \tau_n / \log n \quad (n \geq 2)$$

where

$$(2) \quad \tau_n = \sum_{k=0}^n \frac{s_k}{k+1} \quad (n \geq 0).$$

If $\lim_{n \rightarrow \infty} t_n = \mu$ exists we say that $\{s_n\}$ is summable by the logarithmic method l . Let Γ denote the family of summability methods consisting of all those Euler methods E_α , Taylor methods T_α , Meyer-König methods S_α , Valiron methods F_α and the Borel exponential method B —the “circle family” (Kreisverfahren)—which are permanent (= regular) and strictly stronger than convergence. For the definitions and basic properties of the various summability methods, see Hardy [1], Meyer-König [3] and Zeller and Beekmann [5].

For any sequence $\{x_n\}$, the symbols $x_n = O_L(n^\alpha)$, $x_n = O(n^\alpha)$ and $x_n = o(n^\alpha)$ will denote, as usual, that $\liminf n^{-\alpha} x_n > -\infty$, $\limsup |n^{-\alpha} x_n| < \infty$ and $\lim n^{-\alpha} x_n = 0$, respectively; the C_1 -transform of $\{x_n\}$ will be denoted by $\{x_n^1\}$, so that $x_n^1 = (x_0 + \dots + x_n)/(n+1)$ for $n \geq 0$. Finally, throughout this note, the sequences labelled $\{s_n\}$, $\{t_n\}$ and $\{\tau_n\}$ are assumed to be related by equations (1) and (2).

2. Theorems.

Theorem 1. (a) Let $\{s_n\}$ be a sequence such that there exists a number μ with

$$(3) \quad t_n \log n = \mu + o(n^{-1/2}).$$

Then $\{s_n\}$ is summable by every member of the family Γ .

In particular, the condition $n^{1/2} t_n \log n = o(1)$ implies that $\{s_n\}$ is sum-

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nable by every member of the family Γ .

(b) In the above, the symbol $o(n^{-1/2})$ cannot be replaced by $O(n^{-1/2})$. Indeed, there exists a C_1 -summable bounded sequence $\{s_n\}$ such that it is not summable by any member of Γ and

$$t_n \log n = O(n^{-1/2}).$$

The last assertion shows that Theorem 1(a) is the best possible in a strong sense.

To prove Theorem 1(a) we use the following

Lemma (Parameswaran [4, Theorem 4]). *If $\{s_n\}$ is a sequence and μ a number with*

$$(s_0 + s_1 + \cdots + s_n)/(n+1) = \mu + o(n^{-1/2}),$$

then $\{s_n\}$ is summable (to μ) by every member of Γ .

{Note. In [4], the Borel method B also should have been mentioned as a member of Γ ; there are also some minor misprints in the proof of Theorem 1.}

Proof of Theorem 1(a). From relation (2) we see that

$$(4) \quad s_n = (n+1)(\tau_n - \tau_{n-1}) \quad \text{for } n \geq 0$$

and hence

$$s_0 + s_1 + \cdots + s_n = (n+1)\tau_n - (0 + \tau_0 + \cdots + \tau_{n-1})$$

or

$$(5) \quad s_n^1 = \tau_n - \theta_n^1 \quad (n \geq 0)$$

where $\theta_n = \tau_{n-1}$ ($n \geq 0$), with $\tau_{-1} = 0$. Writing $u_n = \tau_n - \mu$, $u_{-1} = -\mu$, we see that

$$\begin{aligned} s_n^1 &= (u_n + \mu) - (n+1)^{-1} \sum_{m=0}^n (u_{m-1} + \mu) \\ &= u_n - (n+1)^{-1} \sum_0^n u_{m-1} \\ &= o(n^{-1/2}) + o(n^{-1/2}) \quad [\text{by (3)}] \\ &= o(n^{-1/2}). \end{aligned}$$

Hence the sequence $\{s_n\}$ is summable by every member of Γ , by the Lemma.

Proof of Theorem 1(b). Given an arbitrary sequence $\{x_n\}$, its counting function $\omega(n)$ is defined as follows: $\omega(n)$ is the number of nonzero terms x_{n_ν} with $n_\nu \leq n$. G. Lorentz [2] has shown that if $\Omega(n)$ is an arbitrary positive monotone increasing function and E_α a regular Euler method stronger than convergence (i.e. $0 < \alpha < 1$ and E_α is defined as in [2] or [5]), then E_α will sum every bounded sequence $\{x_n\}$ for which $\omega(n) \leq \Omega(n)$ if and only if $\Omega(n) = o(n^{1/2})$. Hence there exists a bounded sequence $\{x_n\}$ with $\omega(n) =$

$O(n^{1/2})$ and $\{x_n\}$ not E_α -summable. Let $\{x_n\}$ be such a sequence. We have then

$$x_0 + x_1 + \cdots + x_n = O(n^{1/2}) \quad \text{or} \quad x_n^1 = (x_0 + x_1 + \cdots + x_n)/(n+1) = O(n^{-1/2}).$$

We define the sequences $\{\tau_n\}$ and $\{s_n\}$ by setting $\tau_n = x_n^1$ for all $n \geq 0$. Then it follows from (5) that

$$(6) \quad s_n = x_n - \theta_n \quad (n \geq 0).$$

We note that $\{\theta_n\}$, being a translate of $\{\tau_n\} = \{x_n^1\}$ is convergent; and $\{x_n\}$ is bounded and not E_α -summable. Hence we see from (6) that $\{s_n\}$ is bounded and not E_α -summable. Since the members of Γ are all equivalent for bounded sequences (Meyer-König [3, Satz 25]), this means that $\{s_n\}$ is not summable by any member of Γ . This proves the theorem.

Theorem 2. Let $\{s_n\}$ be a sequence and μ a real number such that

$$(7) \quad s_n = O_L(1)$$

and

$$(8) \quad \frac{1}{n} \sum_{k=1}^n k^{1/2} t_k \log k = \mu + o(n^{-1/2})$$

where $\{t_k\}$ is defined by (1) and (2). Then $\{s_n\}$ is summable by every member of Γ .

Proof. Set $\phi_k = \tau_k - \mu k^{-1/2}$ ($k \geq 1$). Then (8) gives us

$$(9) \quad \frac{1}{n} \sum_1^n k^{1/2} \phi_k = \frac{1}{n} \sum_1^n (k^{1/2} \tau_k - \mu) = o(n^{-1/2})$$

and hence $\phi_n = o(1)$; also, the Lemma shows that

$$(10) \quad \text{the sequence } \{k^{1/2} \phi_k\} \text{ is } E_{1/2}\text{-summable to } 0.$$

From relation (4) we have for $n \geq 1$:

$$\begin{aligned} s_n &= (n+1)(\tau_n - \tau_{n-1}) \\ &= \left(\frac{n+1}{n}\right) \{n^{1/2} [n^{1/2} \tau_n - (n-1)^{1/2} \tau_{n-1}] - \left(\frac{n+1}{n}\right) n^{1/2} [n^{1/2} - (n-1)^{1/2}] \tau_{n-1} \\ &= \left(\frac{n+1}{n}\right) n^{1/2} \{n^{1/2} \phi_n + \mu - (n-1)^{1/2} \phi_{n-1} - \mu\} - \left(\frac{n+1}{n}\right) n^{1/2} O(n^{-1/2}) (\phi_{n-1} + \mu n^{-1/2}) \\ &= \left(\frac{n+1}{n}\right) n^{1/2} \Delta(n^{1/2} \phi_n) - \left(\frac{n+1}{n}\right) O(\phi_{n-1}) + O(n^{-1/2}) \end{aligned}$$

[where for any sequence $\{u_n\}$ we define $\Delta u_n = u_n - u_{n-1}$]

$$\begin{aligned} &= \left(\frac{n+1}{n}\right) n^{1/2} \Delta(n^{1/2} \phi_n) + o(1) \quad [\text{by (9)}] \\ &= O_L(1) \quad [\text{by (7)}]. \end{aligned}$$

Hence

$$(11) \quad n^{1/2} \Delta(n^{1/2} \phi_n) = O_L(1).$$

Hence (10) yields, in view of the well-known Tauberian theorem for Euler methods, that the sequence $\{n^{1/2} \phi_n\}$ converges to 0. Thus the sequence $\{n^{1/2} \tau_n - \mu\}$ converges to 0; i.e.

$$(12) \quad \tau_n = \mu n^{-1/2} + o(n^{-1/2}).$$

Now

$$\begin{aligned} (13) \quad s^1 &= \tau_n - (n+1)^{-1} \sum_0^n \tau_k + \tau_n / (n+1) \quad [\text{by (5)}] \\ &= \mu n^{-1/2} + o(n^{-1/2}) - (n+1)^{-1} \sum_1^n [\mu k^{-1/2} + o(k^{-1/2})] \quad [\text{by (12)}] \\ &= \mu n^{-1/2} + o(n^{-1/2}) - \mu(n+1)^{-1} [2n^{1/2} + O(1/n) + o(n^{1/2})] \\ &= -\mu(n+1)^{-1/2} + o(n^{-1/2}). \end{aligned}$$

If we define $\{u_n\}$ by setting $u_n^1 = \mu(n+1)^{-1/2}$ then it is easy to verify that

$$(14) \quad \{u_n\} \text{ is a convergent sequence.}$$

But (13) gives $s_n^1 + u_n^1 = o(n^{-1/2})$ and hence, by the Lemma, the sequence $\{s_n + u_n\}$ is summable by every member of Γ . In view of (14), this proves the theorem.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MANITOBA, WINNIPEG R3T 2N2,
CANADA