UNIQUENESS FOR NONLINEAR CAUCHY PROBLEMS IN BANACH SPACES¹

JEROME A. GOLDSTEIN²

ABSTRACT. Recently Medeiros and Diaz and Weinacht have considered the question of uniqueness for the Cauchy problem for ordinary differential equations in a complex Hilbert space. The present paper extends their results to the case of equations in an arbitrary real or complex Banach space.

1. Introduction. We shall establish uniqueness criteria for the Cauchy problem

$$(1) \quad (d/dt)u(t) = f(t, u(t)),$$

$$(2) \quad u(0) = u_0;$$

here f takes values in a real or complex Banach space X and $u_0 \in X$. Our results generalize similar results recently obtained by L. A. Medeiros [7] and J. B. Diaz and R. J. Weinacht [1], who dealt with the case when X is a complex Hilbert space. Our proof is even infinitesimally simpler than those of the above authors; the key tool is an elementary observation of T. Kato concerning duality maps.

2. Preliminaries. Throughout this paper X is a real or complex Banach space with norm $\|\cdot\|$, and u is an X -valued function on an interval I of real numbers. We use the terminology of Hille and Phillips [5]. u is *strongly absolutely continuous* iff for each compact subinterval $[a, b]$ of I and for each $\epsilon > 0$, there is a $\delta > 0$ such that $\sum_{i=1}^n \|u(b_i) - u(a_i)\| < \epsilon$ whenever $a \leq a_i < b_i \leq b$, $i = 1, \dots, n$, and $\sum_{i=1}^n (b_i - a_i) < \delta$. If u is strongly absolutely continuous, then the real-valued function $\|u(\cdot)\|$ is absolutely continuous.

We state here some elementary sufficient conditions for strong absolute continuity. If u is strongly differentiable a.e. with strong derivative u' , and if $u(t) - u(a) = \int_a^t u'(s) ds$ (Bochner integral) for $a, t \in I$, then u is strongly absolutely continuous. (The converse, while false in general, is true if X is reflexive.) If u has a weakly continuous strong derivative

Received by the editors December 13, 1974.

AMS (MOS) subject classifications (1970). Primary 34G05, 47H15; Secondary 35A05.

¹ This paper was accepted in December, 1971 to appear in *Applicable Analysis*. Unanticipated publication delays resulted in resubmission to this journal.

² Supported by National Science Foundation grant GP-28652.

(everywhere), then $u(t) - u(a) = \int_a^t u'(s) ds$, and u is strongly absolutely continuous.

Let X^* be the antidual of X , i.e., the space of all continuous antilinear (or conjugate linear) functionals on X . The image of $x \in X$ under $\phi \in X^*$ will be denoted by $\langle x, \phi \rangle$. For each $x \in X$ let $J(x)$ be the (nonempty) set of all $\phi \in X^*$ for which $\langle x, \phi \rangle = \|x\|^2 = \|\phi\|^2$. A *duality map* of X is a function $j: X \rightarrow X^*$ such that $j(x) \in J(x)$ for each $x \in X$.

We shall rely on the following simple but useful observation of T. Kato [6, p. 510].

Lemma. *Suppose u has a weak derivative $u'(s) \in X$ at $t = s$, and suppose $\|u(\cdot)\|$ is differentiable at $t = s$. Then*

$$\|u(s)\|(d/dt)\|u(t)\|_{t=s} = \operatorname{Re} \langle u'(s), \phi \rangle$$

for each $\phi \in J(u(s))$.

For completeness we quote Kato's short proof. Since

$$\operatorname{Re} \langle u(t), \phi \rangle \leq \|u(t)\| \|\phi\| = \|u(t)\| \|u(s)\|$$

and

$$\operatorname{Re} \langle u(s), \phi \rangle = \|u(s)\|^2,$$

we have

$$\operatorname{Re} \langle u(t) - u(s), \phi \rangle \leq \|u(s)\|(\|u(t)\| - \|u(s)\|).$$

Dividing both sides by $t - s$ and letting $t \rightarrow s$ from above and from below, we obtain

$$\operatorname{Re} \langle u'(s), \phi \rangle \stackrel{\leq}{\geq} \|u(s)\|(d/dt)\|u(t)\|_{t=s}.$$

3. The uniqueness theorems.

Theorem 1. *Let f have domain D contained in $[0, c) \times X$ and range contained in X . Suppose that there is a duality map j of X such that*

$$(3) \quad \operatorname{Re} \langle f(t, x) - f(t, y), j(x - y) \rangle \leq t^{-1} \|x - y\|^2$$

whenever $t > 0$ and $(t, x), (t, y) \in D$. Then there is at most one solution u of (1), (2) on $[0, c)$ in the following sense:

(i) u is strongly absolutely continuous and has a strong right derivative u' a.e.;

(ii) $u'(t) = f(t, u(t))$ a.e. in $[0, c)$;

(iii) $u(0) = u_0$;

(iv) strong $\lim_{t \rightarrow 0+} t^{-1}(u(t) - u(0)) = u_1$.

(ii) and (iv) together form a slightly weaker condition than the condition

(ii') $u'(t) = f(t, u(t))$ for $t \in [0, c) \setminus N$, where N is a Lebesgue null set in $(0, c)$.

When X is a (real or complex) Hilbert space, $\langle \cdot, \cdot \rangle$ is the inner product on X and j is the identity operator; thus the above result generalizes the corresponding results in [7], [1].

Proof of Theorem 1. The basic outline of the proof is as in [7], [1]. Let u, v be solutions of (1), (2) in the sense of (i)–(iv), with the same u_0, u_1 . We must show $u(t) = v(t)$, $0 \leq t < c$. By Kato's lemma and (3) we have for a.e. $t \in [0, c)$,

$$\begin{aligned} \|u(t) - v(t)\| \|(d/dt)\|u(t) - v(t)\| &= \operatorname{Re} \langle f(t, u(t)) - f(t, v(t)), j(u(t) - v(t)) \rangle \\ &\leq t^{-1} \|u(t) - v(t)\|^2. \end{aligned}$$

Claim. $(d/dt)\|u(t) - v(t)\| \leq t^{-1} \|u(t) - v(t)\|$ a.e. in $[0, c)$.

This is immediate for a.e. t for which $u(t) \neq v(t)$. For a.e. $t \in [0, c)$ for which $u(t) = v(t)$ we have $u'(t) = v'(t)$ by (ii), and

$$\begin{aligned} (d/dt)\|u(t) - v(t)\| &= \lim_{h \rightarrow 0^+} h^{-1} \|u(t+h) - v(t+h)\| \\ &= \lim_{h \rightarrow 0^+} \|h^{-1}(u(t+h) - u(t)) - h^{-1}(v(t+h) - v(t))\| = 0, \end{aligned}$$

and thus our Claim is proved.

From the Claim it follows that $(d/dt)(t^{-1}\|u(t) - v(t)\|) \leq 0$ a.e., and so the function $t \rightarrow t^{-1}\|u(t) - v(t)\|$ is monotone nonincreasing on $(0, c)$. Also,

$$\begin{aligned} \lim_{t \rightarrow 0^+} t^{-1}(u(t) - v(t)) &= \lim_{t \rightarrow 0^+} \{t^{-1}(u(t) - u(0)) - t^{-1}(v(t) - v(0))\} \\ &= u_1 - v_1 = 0. \end{aligned}$$

It follows easily that $u(t) = v(t)$ for all $t \in [0, c)$, and the proof is complete.

We note, following [1], that t^{-1} in (3) cannot be replaced by Mt^{-1} for some $M > 1$. (Cf. Perron [8] for a counterexample.)

The final result is a generalized Osgood type uniqueness theorem. Let $\phi(\cdot, \cdot)$ be a nonnegative measurable function defined on $(0, c) \times [0, \infty)$. Call ϕ a *permissible function* iff whenever w is a nonnegative solution of $w(t) \leq \int_0^t \phi(s, w(s)) ds$ a.e. in $[0, c)$, then $w = 0$ a.e.

Theorem 2. Let f have domain D contained in $[0, c) \times X$ and range contained in X . Let ϕ be a permissible function, and suppose there is a duality map j of X such that

$$\operatorname{Re} \langle f(t, x) - f(t, y), j(x - y) \rangle \leq \|x - y\| \phi(t, \|x - y\|)$$

whenever $t > 0$ and $(t, x), (t, y) \in D$. Then there is at most one solution u of (1), (2) satisfying (i)–(iv) of Theorem 1.

Theorem 1 is the special case of Theorem 2 which corresponds to taking $\phi(t, r) = r/t$. Theorem 2 generalizes results in [7], [1]. For examples of permissible functions see Walter [9, pp. 80–85]. We gratefully acknowledge a suggestion of Hans Weinberger which led to the present formulation of Theorem 2.

Proof of Theorem 2. As in the proof of Theorem 1, let u, v be solutions to (1), (2) satisfying (i)–(iv). Arguing as in that proof we get

$$\begin{aligned}\|u(t) - v(t)\|(d/dt)\|u(t) - v(t)\| &= \operatorname{Re} \langle f(t, u(t)) - f(t, v(t)), j(u(t) - v(t)) \rangle \\ &\leq \|u(t) - v(t)\|\phi(t, \|u(t) - v(t)\|);\end{aligned}$$

and we conclude as before that

$$(d/dt)\|u(t) - v(t)\| \leq \phi(t, \|u(t) - v(t)\|) \quad \text{a.e. in } [0, c].$$

Since ϕ is permissible, it follows that $\|u(t) - v(t)\| = 0$ a.e., and the proof is complete.

4. Concluding remarks. The map j , defined by

$$j(f)(x) = \begin{cases} 0 & \text{if } f(x) = 0; \\ \|f\|_p^{2-p} f(x) |f(x)|^{p-2} & \text{otherwise,} \end{cases}$$

is a duality map for the real or complex space $L^p(\Omega)$ on an arbitrary measure space Ω for $1 \leq p < \infty$. For a brief discussion of duality maps in Orlicz spaces, see [4].

The techniques in this paper can be used to extend other results concerning ordinary differential equations in a complex Hilbert space to an arbitrary Banach space setting. For instance, in Flett's paper [3], Theorems 2 and 4 (on pp. 336, 341) hold if, in Flett's notation, Y is a real or complex Banach space and $\langle f(t, y) - f(t, z), y - z \rangle$ is replaced by $\langle f(t, y) - f(t, z), j(y - z) \rangle$, where j is a duality map of Y .

New uniqueness theorems for the Euler-Poisson-Darboux equation and other singular linear equations can be obtained as an application of the results of this paper. These results will appear elsewhere (cf. Donaldson-Goldstein [2]).

REFERENCES

1. J. B. Diaz and R. J. Weinacht, *On nonlinear differential equations in Hilbert spaces*, *Applicable Anal.* 1 (1971), 31–41. MR 43 #7746.
2. J. A. Donaldson and J. A. Goldstein (in preparation).
3. T. M. Flett, *Some applications of Zygmund's lemma to nonlinear differential equations in Banach and Hilbert spaces*, *Studia Math.* 44 (1972), 335–344, 649–650.
4. J. A. Goldstein, *Groups of isometries on Orlicz spaces*, *Pacific J. Math.* 48 (1973), 387–393.
5. E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 31, Amer. Math. Soc., Providence, R. I., 1957. MR 19, 664.

6. T. Kato, *Nonlinear semigroups and evolution equations*, J. Math. Soc. Japan 19 (1967), 508–520. MR 37 #1820.
7. L. A. Medeiros, *On nonlinear differential equations in Hilbert spaces*, Amer. Math. Monthly 76 (1969), 1024–1027. MR 40 #1685.
8. O. Perron, *Eine hinreichende Bedingung für die Unität der Lösung von Differentialgleichungen erster Ordnung*, Math. Z. 28 (1928), 216–219.
9. W. L. Walter, *Differential- und Integral-Ungleichungen und ihre Anwendung bei Abschätzungs- und Eindeutigkeits-Problemen*, Springer Tracts in Natural Philosophy, vol. 2, Springer-Verlag, Berlin and New York, 1964; English transl., *Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 55*, Springer-Verlag, New York and Berlin, 1970. MR 30 #2302; 42 #6391.

DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, NEW ORLEANS, LOUISIANA 70118