SUBSPACES OF $LC(H)$ AND C_p

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ABSTRACT. This paper deals with certain properties of the subspaces of $LC(H)$ and C_p , and namely those connected with the reflexivity and with the property of containing classical spaces. It is proved that any subspace of C_p ($1 \leq p < \infty$) is either isomorphic to Hilbert space or it contains a subspace isomorphic to l_p . For C_1 and $LC(H)$ the same results were obtained by J. R. Holub, cf. [4].

Introduction. Let H be a complex Hilbert space. Denote by $LC(H)$ or C_∞ the space of all compact operators on H , with the usual operator norm. Let C_p ($1 \leq p < \infty$) be the space of all operators $T: H \rightarrow H$ for which $|T|_p^p = \text{trace}(T^*T)^{p/2} < \infty$ with the norm $|\cdot|_p$. $LC(H)$ and C_p are Banach spaces.

The word "subspace" in this paper will mean "closed subspace". For the subspaces of $LC(H)$ and C_p it is known that [1] if X is a subspace of $LC(H)$ which is complemented in the space $L(H)$ of all bounded operators on H , then X is a reflexive space. J. R. Holub [4] proved that if a subspace X of $LC(H)$ or C_1 is isomorphic to H , then it is complemented in $L(H)$, and if X is not isomorphic to H it contains a subspace isomorphic to c_0 or l_1 respectively. We will prove results similar to those of Holub for the C_p spaces ($1 \leq p < \infty$).

We shall denote an orthogonal projection from H onto a subspace E by P_E .

For any two subspaces E, F of H and any $1 \leq p \leq \infty$, R_p^{EF} will denote the subspace of C_p :

$$R_p^{EF} = \{A \in C_p : (1 - P_F)A(1 - P_E) = 0\}.$$

This subspace is closed in the norm $|\cdot|_p$.

Proposition 1. *If E and F are finite dimensional subspaces of H , then R_p^{EF} is isomorphic to a Hilbert space for any $1 \leq p \leq \infty$.*

Proof. It is easily seen that $R_\infty^{EF} \subset R_1^{EF}$.

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Since $|A|_p \geq |A|_q$ for $1 \leq p \leq q \leq \infty$ and every operator A , we get that if A belongs to some R_p^{EF} , $1 \leq p \leq \infty$, then it also belongs to all R_q^{EF} for $1 \leq q \leq \infty$. Thus R_p^{EF} is isomorphic to R_2^{EF} for any $1 \leq p \leq \infty$. But R_2^{EF} is a subspace of a Hilbert space C_2 . Hence the result follows.

Lemma 1. Let $1 \leq p \leq \infty$, $\{M_i\}_{i=1}^\infty$ and $\{N_i\}_{i=1}^\infty$ be two orthogonal sequences of subspaces, i.e. M_i is orthogonal to M_j and N_i is orthogonal to N_j for any $i \neq j$. Suppose that $\{A_i\}_{i=1}^\infty$ is a sequence of operators of norm $|A_i|_p = 1$ such that $|A_i(1 - P_{M_i})|_p = 0$, $|(1 - P_{N_i})A_i|_p = 0$ for $i = 1, 2, 3, \dots$.

Suppose that $Y = \overline{\text{span}}\{A_i\}_{i=1}^\infty$, where the closure is taken in the C_p norm. Then Y is isometric to l_p (c_0 when $p = \infty$).

Proof. Consider an operator $A = \sum_{i=1}^N a_i A_i$, $\{a_i\}$ being scalars. It will suffice to show that $|A|_p = (\sum_{i=1}^N |a_i|^p)^{1/p}$.

For every $i = 1, 2, \dots, N$, $A_i(u) = \sum_{j=1}^{m_i} \lambda_{ij}(u, e_{ij})\phi_{ij}$, where $u \in H$, $\{\lambda_{ij}\}_{j=1}^{m_i}$ are the characteristic numbers of $(A_i A_i^*)^{1/2}$, $\{e_{ij}\}_{j=1}^{m_i}$ and $\{\phi_{ij}\}_{j=1}^{m_i}$ are orthonormal sequences; cf. [6]. Then

$$A(u) = \sum_{i=1}^N \sum_{j=1}^{m_i} a_i \lambda_{ij}(u, e_{ij})\phi_{ij}$$

where $\{e_{ij}\}_{ij}$ and $\{\phi_{ij}\}_{ij}$ are orthonormal sequences.

It is known that the C_p norm of such an operator [3] is equal to

$$|A|_p = \left(\sum_{i=1}^N \sum_{j=1}^{m_i} |a_i \lambda_{ij}|^p \right)^{1/p} = \left(\sum_{i=1}^N |a_i|^p |A_i|_p^p \right)^{1/p} = \left(\sum_{i=1}^N |a_i|^p \right)^{1/p}.$$

We shall need the following result which was proved in [5].

Proposition 2. Let $A \in C_p$ ($1 \leq p \leq \infty$), $\delta > 0$. Then there is a finite dimensional subspace M of H such that $|A(1 - P_M)|_p < \delta$.

Definition. Let $1 \leq p \leq \infty$. If X is a subspace of C_p such that there are two finite dimensional subspaces E and F with the property that the projection $Q^{EF}(A) = P_F A + A P_E - P_F A P_E$ from C_p onto R_p^{EF} is an isomorphism on X , then we say that X satisfies the "finite range property".

Lemma 2. If X is a subspace of C_p which does not satisfy the finite range property, then for any two finite dimensional subspaces E and F of H and any $\delta > 0$, there are two finite dimensional subspaces M and N and an operator $A \in X$ with $|A|_p = 1$ such that:

1. M is orthogonal to E ;
2. N is orthogonal to F ;
3. $|A(1 - P_M)|_p < \delta$;
4. $|(1 - P_N)A|_p < \delta$.

Proof. By assumption, $Q^{EF}: X \rightarrow R_p^{EF}$ is not an isomorphism. So there is an operator A in X such that $|A|_p = 1$ and

$$|Q^{EF}(A)|_p = |P_F A + A P_E - P_F A P_E|_p < \delta/2.$$

Observe that $P_F A + A P_E - P_F A P_E = A P_E + P_F A(1 - P_E)$; hence

$$(1) \quad |A P_E|_p = |(A P_E + P_F A(1 - P_E)) P_E|_p \leq |A P_E + P_F A(1 - P_E)|_p < \delta/2.$$

In the same way we derive that

$$(2) \quad |P_F A|_p < \delta/2.$$

By Proposition 2 there is a finite dimensional subspace $\tilde{M}$ in H such that

$$(3) \quad |A(1 - P_{\tilde{M}})|_p < \delta/2.$$

Take M to be the projection of $\tilde{M}$ on the orthocomplement of E , that is

$$(4) \quad M = (1 - P_E)\tilde{M}.$$

We will prove that the norm of $A(1 - P_M)$ is less than δ .

$$|A(1 - P_M)|_p \leq |A P_E|_p + |A(1 - P_M - P_E)|_p.$$

From (1), $|A P_E|_p$ is less than $\delta/2$. The subspace $\tilde{M}$ belongs to $\text{span}(M \cup E)$ by (4), so that the orthocomplement of $\text{span}(M \cup E)$ belongs to the orthocomplement of $\tilde{M}$. Thus

$$\begin{aligned} |A(1 - P_M - P_E)|_p &= |A(1 - P_{\tilde{M}})(1 - P_M - P_E)|_p \\ &\leq |A(1 - P_{\tilde{M}})|_p \cdot |(1 - P_M - P_E)|_\infty < \delta/2. \end{aligned}$$

We take $N = (1 - P_F)A(\tilde{M})$. It is obvious that N is finite dimensional and orthogonal to F . It remains only to prove that $|(1 - P_N)A|_p$ is less than δ . Now,

$$\begin{aligned} |(1 - P_N)A|_p &\leq |(1 - P_N)(1 - P_F)A P_{\tilde{M}}|_p + |(1 - P_N)P_F A P_{\tilde{M}}|_p \\ &\quad + |(1 - P_N)A(1 - P_{\tilde{M}})|_p. \end{aligned}$$

From the definition of N , it follows that $(1 - P_N)(1 - P_F)A P_{\tilde{M}} = 0$. Using (2) and (3) we derive $|(1 - P_N)A|_p < \delta$. This completes the proof.

Theorem 1. Let X be a subspace of C_p ($1 \leq p \leq \infty$) which does not satisfy the finite range property. Then for any $\epsilon > 0$, there is a subspace Y of X which is ϵ isomorphic to l_p (or c_0 when $p = \infty$), and there is a projection of norm less than $1 + \epsilon$ from C_p onto this Y .

Proof. Without loss of generality, take $\epsilon < 2$. Take an operator $\tilde{A}_1$

from X with the norm $|\tilde{A}_1|_p = 1$. From Proposition 2 it follows that there is a finite dimensional subspace M_1 such that $|\tilde{A}_1(1 - P_{M_1})|_p < \epsilon/32$. Let $N_1 = \tilde{A}_1(M_1)$; then

$$|(1 - P_{N_1})\tilde{A}_1|_p \leq |(1 - P_{N_1})\tilde{A}_1(1 - P_{M_1})|_p < \epsilon/32.$$

Construct by induction, with the aid of Lemma 2, a sequence of operators $\tilde{A}_i$ which belong to X , and sequences of orthogonal subspaces $\{M_i\}_{i=1}^\infty$ and $\{N_i\}_{i=1}^\infty$ of H such that

$$(5) \quad |\tilde{A}_i|_p = 1,$$

$$(6) \quad |\tilde{A}_i(1 - P_{M_i})|_p < \epsilon/2^{i+4},$$

$$(7) \quad |(1 - P_{N_i})\tilde{A}_i|_p < \epsilon/2^{i+4}.$$

Define a subspace $\tilde{Y} = \overline{\text{span}}\{\tilde{A}_i\}_{i=1}^\infty$ with closure taken in the C_p norm. Consider for each i the operator:

$$(8) \quad A_i = P_{N_i}\tilde{A}_iP_{M_i}/|P_{N_i}\tilde{A}_iP_{M_i}|_p.$$

The space $Y = \overline{\text{span}}\{A_i\}_{i=1}^\infty$ satisfies the conditions of Lemma 1 and therefore is isometric to l_p (or c_0 when $p = \infty$). To see that $\tilde{Y}$ is ϵ -isomorphic to Y it is sufficient to show that

$$(9) \quad \left| \sum_{i=1}^N a_i \tilde{A}_i - \sum_{i=1}^N a_i A_i \right|_p < \frac{\epsilon}{4} \left(\sum_{i=1}^N |a_i|^p \right)^{1/p}.$$

This follows from (5)–(8). It remains only to prove that there is a projection with norm less than $(1 + \epsilon)$ from C_p onto $\tilde{Y}$.

For each i , let $\tilde{Q}_i$ be defined by $\tilde{Q}_i(A) = P_{N_i}AP_{M_i}$; let $\tilde{P}_i$ be a norm one projection from the image of $\tilde{Q}_i$ on the one-dimensional space spanned by A_i , and define $\tilde{P}$ by

$$\tilde{P}(A) = \sum_i \tilde{P}_i \tilde{Q}_i(A).$$

By using Lemma 1 we derive that

$$|\tilde{P}(A)|_p^p = \sum_i |\tilde{P}_i \tilde{Q}_i(A)|_p^p \leq \sum_i |\tilde{Q}_i(A)|_p^p.$$

The C_p norm of an operator T is equal to

$$|T|_p = \sup \left\{ \sum_\alpha |(T\phi_\alpha, \psi_\alpha)|^p \right\}^{1/p}$$

where ϕ_α and ψ_α run over all pairs of orthonormal bases of H [5]. It follows

that $\sum_i |Q_i(A)|_p^p < |A|_p^p$ and then $|\tilde{P}| = 1$. From (9) it follows that the operator $\tilde{P}$, which is the restriction of P to the subspace $\tilde{Y}$, satisfies

$$\left| \tilde{P} \sum_{i=1}^N a_i \tilde{A}_i \right|_p \geq \left| \tilde{P} \sum_{i=1}^N a_i A_i \right|_p - \left| \tilde{P} \left(\sum_{i=1}^N a_i \tilde{A}_i - \sum_{i=1}^N a_i A_i \right) \right|_p > \left(1 - \frac{\epsilon}{2} \right) |A|_p.$$

Then the norm of $\tilde{p}^{-1} \tilde{P}$ is less than $1 + \epsilon$. Taking $p = \tilde{p}^{-1} \tilde{p}$, we get a projection from C_p on $\tilde{Y}$ of norm less than $1 + \epsilon$.

Theorem 2. *Let X be a subspace of C_p ($1 \leq p \leq \infty$), $p \neq 2$. X satisfies the finite range property if and only if X is isomorphic to a Hilbert space.*

Proof. Assume that X satisfies the finite range property. Then there are two finite dimensional subspaces E and F such that the function Q^{EF} is an isomorphism from X onto a subspace of R_p^{EF} . Using Proposition 1, we obtain the desired result.

Let X be isomorphic to a Hilbert space. It is obvious that X does not contain subspaces isomorphic to c_0 or l_p for $p \neq 2$. Then from Theorem 1 it follows that X satisfies the finite range property.

Proposition 3. *Let X be a subspace of C_p which satisfies the finite range property. Then there is a bounded projection from C_p onto X .*

Proof. From the definition of X , there are two finite dimensional subspaces E and F of H such that $\tilde{Q}^{EF}$, the restriction of Q^{EF} to X , is an isomorphic mapping from X onto a subspace Z of R_p^{EF} . As Z is also isomorphic to a Hilbert space, there is a bounded projection P from R_p^{EF} onto Z . Then $P = (\tilde{Q}^{EF})^{-1} \tilde{P} \cdot Q^{EF}$ is a bounded projection from C_p onto X .

Corollary. *Let X be a subspace of $LC(H) = C_\infty$, the space of compact operators on a Hilbert space H . The following conditions are equivalent:*

1. X is reflexive.
2. There is an $\epsilon > 0$ such that there exists no complemented subspace in X which is ϵ -isomorphic to c_0 .
3. X satisfies the finite range property.
4. X is isomorphic to a Hilbert space.
5. X is complemented in the space of all bounded operators $L(H)$.

Proof. $1 \Rightarrow 2$ is obvious.

$2 \Rightarrow 3$ by Theorem 1.

$3 \Rightarrow 4$ by Theorem 2.

$4 \Rightarrow 5$ is proved by J. R. Holub [4].

$5 \Rightarrow 1$ is proved by Ch. A. Akemann [1].

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