COMMON FIXED POINTS OF COMMUTING MAPPINGS

WILLIAM J. GRAY AND CAROL M. SMITH

ABSTRACT. Let X be a dendroid and S an abelian semigroup of continuous monotone self-mappings of X . A point $x \in X$ is fixed under S if $g(x) = x$ for all $g \in S$. Let $f: X \rightarrow X$ be continuous and commute with each element of S . It is shown that f and S have a common fixed point.

In what follows, a continuum will be a compact connected Hausdorff space. A continuum X is hereditarily unicoherent if any two subcontinua of X meet in a continuum; if, in addition, X is locally connected (arcwise connected) and metrizable, then X is called a dendrite (dendroid). If S is an abelian semigroup (under composition) of continuous self-mappings of X , a point $x \in X$ is fixed under S if $g(x) = x$ for all $g \in S$. A map $f: X \rightarrow X$ is monotone if $f^{-1}(x)$ is connected for all $x \in X$.

For some time there was an open conjecture, due to Dyer, Isbell, and Shields, that an abelian semigroup of continuous self-mappings of a dendrite X has a fixed point. However, Huneke [5], and Boyce [8] constructed a pair of commuting continuous self-mappings of the unit interval which have no common fixed point. Thus to obtain common fixed points of commuting mappings of hereditarily unicoherent continua, it is necessary to assume more than continuity. Among the results obtained to date are (1) Cohen [1]: a pair of open commuting self-mappings of the unit interval have a common fixed point (the "full" mappings of [1] are actually open mappings), and (2) Gray [2]: an abelian semigroup of continuous monotone self-mappings of a hereditarily unicoherent, hereditarily decomposable continuum X leaves a point of X fixed. (This is a generalization of Hamilton's theorem [4].) In this paper we show that if S is an abelian semigroup of continuous monotone self-mappings of a dendroid X , and if $f: X \rightarrow X$ is continuous and commutes with each element of S , then f and S have a common fixed point.

If X is a dendroid and $a, b \in X$, there is a unique arc $A(a, b) \subset X$ from a to b . Choose $e \in X$ and for $x, y \in X$ define $x \leq y$ if and only if $x \in A(e, y)$. By Ward [6], we have:

(1) $\leq$ is a partial order with least element e , called the arc order on X with root e .

Received by the editors August 28, 1974.

AMS (MOS) subject classifications (1970). Primary 54H25; Secondary 54F20.

Key words and phrases. Dendroid, abelian semigroup, monotone mapping, open mapping, arc order, hereditarily unicoherent continuum.

Copyright © 1975. American Mathematical Society

(2) If $x \in X$, then $[e, x] = \{y \in X: e \leq y \leq x\} = A(e, x)$ is a chain under $\leq$.

(3) Each nonempty chain in X has a supremum, and each subcontinuum has a least element.

Let $f: X \rightarrow X$ be a monotone continuous surjection which leaves e fixed. Define $f^*: X \rightarrow X$ by $f^*(x) = \text{Inf } f^{-1}(x)$. Then we have

(4) If $x, y \in X$, then $f(A(x, y)) = A(f(x), f(y))$; hence f is order preserving (Ward [7]).

By Gray-Nelson [3] we have

(5) f^* is order preserving.

(6) $f \cdot f^* = \text{identity on } X$.

(7) If $g: X \rightarrow X$ is continuous and commutes with f , then $gf^{-1}(x) \subset f^{-1}g(x)$ implies $f^*g(x) \leq gf^*(x)$, for all $x \in X$.

If $x \in X$ we write $M(x) = \{y \in X: x \leq y\}$. As usual, $x < y$ means $x \leq y$ and $x \neq y$. Note that $A(x, y)$ is actually the intersection of all subcontinua of X containing x and y ; in particular, if H is a subcontinuum of X containing both e and x , then $[e, x] \subset H$.

Theorem. *Let S be an abelian semigroup of continuous monotone self-mappings of a dendroid X . Let $f: X \rightarrow X$ be a continuous mapping which commutes with each element of S . Then f and S have a common fixed point.*

Proof. By Zorn's lemma, there is a subcontinuum A of X which is minimal with respect to being nonempty and being mapped into itself by each element of $S \cup \{f\}$. Since each element of $S \cup \{f\}$ maps A onto itself, we may thus assume without loss of generality that each element of S is surjective.

Suppose to the contrary that f and S have no common fixed point. By [2], we may choose a point $e \in X$ which is fixed under S . Let $\leq$ be the arc order on X with root e , so that each element of S is order preserving. Define

$$J = \{x \in X: x \text{ is fixed under } S\},$$

$$K = \{x \in X: f(x) \in M(x)\}, \text{ and}$$

$$L = J \cap K.$$

Since $M(e) = X$, we have $e \in L$.

Let C be a maximal chain in L and $p = \text{Sup } C$. We wish to prove that $p \in L$. Since J is closed and $p \in \overline{C} \subset J$, we need only show that $p \in K$. Suppose $p \notin K$ so that $p \notin [e, f(p)]$. Since we have $p \neq f(p)$ and f is continuous, we may find y with $e < y < p$ for which

$$(8) \quad f([y, p]) \cap [y, p] = \emptyset.$$

Let $u = \text{Inf } f([y, p])$; then, of course, $u \in [e, f(p)]$. For each $x \in C \cap [y, p]$ we have $u \leq f(x)$ and $x \leq f(x)$ so that x and u compare under $\leq$ (use (2) above). If we had $u < x$ for some such x , then by (8), $H = [e, u] \cup f([y, p])$

would be a continuum containing e and $f(x)$ but not x . We would then have $[e, f(x)] \subset H$, contrary to $x \leq f(x)$. Thus $x \leq u$ for all $x \in C \cap [y, p]$, and so $p \leq u$. But then by definition of u , we find $p \leq f(p)$. This contradiction proves that $p \in K$, hence $p \in L$, and p is a maximal element of L .

Since f commutes with each element of S , we have $f(p) \in J$, and since we have assumed that f and S have no common fixed point, this means $p < f(p)$, and $f(p) \notin K$. Let $M = K \cap [p, f(p)]$, and let $q = \text{Sup } M$. We prove that $p < q$: in fact we may choose $y \in X$ with $p < y < f(p)$ for which $[p, y] \cap f([p, y]) = \emptyset$. Let $z = \text{Inf } [e, f(p)] \cap f([p, y])$. Since X is hereditarily unicoherent, $H = [e, f(p)] \cap f([p, y])$ is a continuum, hence is either an arc with endpoints z and $f(p)$, or is $\{f(p)\}$. Since $y \notin H$, we conclude that $y < z \leq f(y)$, hence $y \in M$. This shows that $p < q$, as asserted.

An argument similar to the one we used above to show that $p \in K$ can be used to show that $q \in M$. As observed previously, $f(p) \notin K$, so

$$(9) \quad p < q < f(p).$$

If $q \in J$, then we would have $q \in L$, which, with (9), would contradict the maximality of p . Thus there is a $g \in S$ for which $g(q) \neq q$.

Now since g is monotone and surjective, and g leaves both p and $f(p)$ fixed, by (4), we have

$$(10) \quad g([p, f(p)]) = [p, f(p)].$$

Thus $g(q) \in [p, f(p)]$. If we assume that $q < g(q)$, we apply g to both sides of the inequality $q \leq f(q)$ to obtain $g(q) \leq g f(q) \leq f g(q)$, i.e. $g(q) \in M$. By (9), the maximality of q is contradicted. Thus

$$(11) \quad g(q) < q.$$

From (10), we conclude that $g^{-1}(q) \cap [e, f(p)] \neq \emptyset$, hence $g^*(q) \in [e, f(p)]$, where g^* is the mapping of (5)–(7).

We assert that the inequality $g^*(q) \leq q$ cannot hold: for otherwise we would have, using (6), $q = g g^*(q) \leq g(q)$, which contradicts (11). Then since $[e, f(p)]$ is a chain, we have $q < g^*(q) \leq f(p)$. We now apply g^* to both sides of the inequality $q \leq f(q)$ and use (5) and (7) to obtain $g^*(q) \leq g^* f(q) \leq f(g^*(q))$, i.e. $g^*(q) \in M$. Once again by (9), the maximality of q is contradicted.

Our only remaining recourse is to admit that f and S must indeed have a common fixed point.

Remark. The theorem of this paper was proved for dendroids for convenience in referencing the literature. It is also true for arboroids, i.e. hereditarily unicoherent continua in which any two points $x, y \in X$ lie in a subcontinuum $A(x, y) \subset X$ which has x and y as its only noncutpoints.

In view of the result of Cohen which was referenced in the introduction

above, the following problem would appear to be of interest:

Question. Let X be a dendrite and S be an abelian semigroup of continuous open surjections of X onto itself. Must S have a fixed point?

REFERENCES

1. Haskell Cohen, *On fixed points of commuting functions*, Proc. Amer. Math. Soc. **15** (1964), 293–296. MR 32 #1692.
2. William J. Gray, *A fixed-point theorem for commuting monotone functions*, Canad. J. Math. **21** (1969), 502–504. MR 39 #942.
3. William J. Gray and James Nelson, Jr., *Commuting monotone mappings of dendroids*, Rocky Mountain J. Math. (to appear).
4. O. H. Hamilton, *Fixed points under transformations of continua which are not connected in Kleinen*, Trans. Amer. Math. Soc. **44** (1938), 18–24.
5. John Philip Huneke, *Two commuting continuous functions from the closed unit interval onto the closed unit interval without a common fixed point*, Topological Dynamics (Sympos., Colorado State Univ., Ft. Collins, Colo., 1967), Benjamin, New York, 1968, pp. 291–298. MR 39 #1609.
6. L. E. Ward, Jr., *Characterization of the fixed point property for a class of set-valued mappings*, Fund. Math. **50** (1961/62), 159–164. MR 24 #A2956.
7. ———, *Monotone surjections having more than one fixed point*, Rocky Mountain J. Math. **4** (1974), 95–106.
8. W. M. Boyce, *Commuting functions with no common fixed point*, Trans. Amer. Math. Soc. **137** (1969), 77–92. MR 38 #4627.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALABAMA—TUSCALOOSA, UNIVERSITY, ALABAMA 35486 (Current address of W. J. Gray)

Current address (C. M. Smith): Department of Mathematics, Birmingham Southern College, Birmingham, Alabama 35204