BRANCHINGS AND PARTITIONS

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ABSTRACT. A generating function is obtained for the number of partitions corresponding to a complete branching on a nonincreasing sequence of n integers. Complete branchings are shown to be related to certain types of plane partitions.

Given a sequence $k_1 \geq k_2 \geq k_3 \geq \cdots \geq k_n$ of integers, a *branching* is a sequence of integers $k'_1, k'_2, \dots, k'_{n-1}$ such that $k_i \geq k'_i \geq k_{i+1}$ ($i = 1, 2, \dots, n-1$). Successively branching $n-1$ times, one obtains a single integer. Such a sequence of $n-1$ successive branchings is called a *complete branching*. S. Gelbart [3] proposed as a Monthly problem that the number of distinct complete branchings of the sequence $k_1, k_2, \dots, k_n$ is equal to

$$(1) \quad \prod_{1 \leq i < j \leq n} [(k_i - k_j + j - i)/(j - i)].$$

A complete branching may be indicated by the triangular array

$$\begin{array}{ccccccc} k_1 & k_2 & k_3 & \cdots & k_n \\ & k'_1 & k'_2 & \cdots & k'_{n-1} \\ T_n: & & k''_1 & \cdots & k''_{n-2} \\ & & & \cdots & \\ & & & & k_1^{(n-1)} \end{array}$$

where $k_i^{(0)} = k_i$ and

$$(2) \quad k_i^{(j)} \geq k_i^{(j+1)} \geq k_{i+1}^{(j)} \quad (i = 1, 2, \dots, n-j-1).$$

For a given integer m and fixed sequence $k_1 \geq k_2 \geq \cdots \geq k_n$, let $P_n(m) = P(m; k_1, \dots, k_n)$ denote the number of arrays T_n satisfying (2) and whose entries sum to m , i.e.,

$$m = \sum_{j=0}^{n-1} \sum_{i=1}^{n-j} k_i^{(j)}.$$

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We shall be interested in evaluating the generating function

$$(3) \quad N(k_1, \dots, k_n; x) = N(x) = \sum_{m=-\infty}^{\infty} P_n(m) x^m.$$

Note that the sum (3) is actually finite.

There is no loss of generality in supposing that the k_i 's are all positive, for this amounts to adding some constant c to each entry of T_n and therefore multiplying (3) by $x^{c(n+1)/2}$. Assuming therefore that each $k_i > 0$, rewrite T_n as

$$\begin{array}{ccccccc} k_1 & k'_1 & k''_1 & \dots & k_1^{(n-1)} & & \\ & k_2 & k'_2 & \dots & k_2^{(n-2)} & & \\ & & & \dots & & & \\ & & & & & & k_n \end{array}$$

The i th row satisfies $k_i \geq k'_i \geq \dots \geq k_i^{(n-i)} > 0$ and therefore may be regarded as a partition of $k_i + k'_i + \dots + k_i^{(n-i)}$. Take the conjugate partition [4, p. 274] of each row and left justify, giving a new array

$$\begin{array}{ccc} j_1 & j'_1 & \dots \\ j_2 & j'_2 & \dots \\ & \dots & \\ j_n & j'_n & \end{array}$$

where $(j_i, j'_i, \dots)$ is the conjugate partition to $(k_i, k'_i, \dots)$.

The resulting array of j 's is always a column-strict plane partition of shape $(k_1, k_2, \dots, k_n)$ and largest part n , as defined in [7, §1]. The sum of the j 's is equal to the sum of the k 's. This correspondence between the array of k 's (complete branchings with first row $k_1, k_2, \dots, k_n$ and sum m) and j 's (column-strict plane partitions of shape $\lambda = (k_1, k_2, \dots, k_n)$ with largest part n and sum m) is easily seen to be a bijection. Hence $P_n(m)$ is equal to the number of column-strict plane partitions of shape λ with largest part n and sum m . In this context, the generating function (3) has been determined (implicitly) by D. E. Littlewood [5, p. 124, Theorem I] and more explicitly in [7, Theorem 15.3]. The result may be stated as follows.

Theorem 1. *We have*

$$(4) \quad N(x) = x^a \prod \frac{(n+j-i)}{(h_{ij})},$$

where $(d) = 1 - x^d$, $a = \sum i k_i$, and $h_{ij} = k_i + \bar{k}_j - i - j + 1$. Here $(\bar{k}_1, \bar{k}_2, \dots)$

is the conjugate partition to $(k_1, k_2, \dots)$, and the product is over all $k_1 + k_2 + \dots + k_n$ pairs (i, j) such that $k_i > 0$ and $\bar{k}_j > 0$.

Some elementary manipulation shows that the right-hand side of (4) can be written in the form

$$(5) \quad N(x) = \prod_{1 \leq i < j \leq n} (x^{k_j} - x^{k_i + j - i}) / (1!) (2!) \cdots (n-1)!$$

where $(i)! = (1)(2) \cdots (i)$. If we put $x = 1$ in (5) we obtain Gelbart's result (1). In the context of plane partitions this result was first given by MacMahon [6].

We shall give a brief indication of how (5) can be proved directly. Clearly one has

$$(6) \quad N(k_1, \dots, k_{n+1}; x) = \sum x^{k'_1 + \dots + k'_n} N(k'_1, \dots, k'_n; x),$$

where the summation is over all $k'_1, \dots, k'_n$ such that $k_1 \geq k'_1 \geq k_2 \geq \dots \geq k'_n \geq k_{n+1}$. But the $n \times n$ determinant

$$(7) \quad D(k_1, \dots, k_n; x) = (-1)^{\binom{n}{2}} x^{n^3 - n} \left| \begin{pmatrix} k_j - j - 1 \\ i - 1 \end{pmatrix} \right|.$$

where $\begin{pmatrix} i \\ j \end{pmatrix} = (i)!/(j)!(i-j)!$, also satisfies the recursion (6) and equals $N(k_1, \dots, k_n; x)$ in the case $n = 1$. It follows that $N(k_1, \dots, k_n; x) = D(k_1, \dots, k_n; x)$. The determinant in (7) may be evaluated by Vandermonde's theorem or otherwise, yielding (5).

We remark that an evaluation of a determinant equivalent to (7) appears in [1]. See also [7, §15] for a direct proof that this determinant (actually a related but equivalent determinant) is given by (4). When $x = 1$, the determinant (7) was first evaluated by Frobenius [2].

It follows from (5) that $N(x) = x^A N(1/x)$, where $A = (n+1) \sum_{i=1}^n k_i$. From this we obtain the symmetry relation $P_n(m) = P_n(A - m)$. It would be interesting to know, whether for fixed $n, k_1, \dots, k_n$, the sequence $\{P_n(m)\}$ is unimodal, that is, whether there exists a C such that $P_n(m-1) \leq P_n(m)$ if $m \leq C$ and $P_n(m) \geq P_n(m+1)$ if $m \geq C$.

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