THE PICARD SEQUENCE OF A FIBRATION

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ABSTRACT. A fibration, in the Zariski topology, of algebraic varieties leads to an exact sequence of cohomology of the relative units functor. Under suitable hypotheses, the exact sequence can be interpreted as a sequence including the Picard groups of the base variety, the total space, and the fibre.

Let $X \rightarrow Y$ be a fibration with fibre F of algebraic varieties (over an algebraically closed field) which is locally trivial in the Zariski topology on Y . If Y and F are smooth and F is rational, Fossum and Iverson [2, Proposition 5, p. 8] constructed an exact sequence relating $\text{Pic}(Y)$, $\text{Pic}(X)$, and $\text{Pic}(F)$. The purpose of the present note is to interpret that sequence in terms of the cohomology, in the Zariski topology, of the relative units sheaf. Besides explaining the Fossum-Iverson sequence, this interpretation helps isolate the hypotheses for the existence of the sequence, which leads to some mild generalizations of the original result.

We fix throughout the following conventions: k denotes an algebraically closed field and all varieties considered are over k and irreducible. Unless otherwise specified, all sheaves are for the Zariski topology, and all sheaf cohomology is in that topology. If V is a variety, U an open subset of V and $\mathcal{F}$ a sheaf on V , then $(\mathcal{F})|_U$ denotes $\mathcal{F}$ restricted to U . If A is an abelian group then $(A)_V$ denotes the constant sheaf on V associated to A . We recall that the higher cohomology groups of constant sheaves on V are trivial, and, hence by descent, that all locally constant sheaves on V are constant. $G_{m,V}$ denotes the units sheaf on V .

Definition 1. Let V be a variety. $U_{k,V}$ denotes the presheaf cokernel of $((k^*)_V \rightarrow G_{m,V})$. We refer to $U_{k,V}$ as the *sheaf of relative units on V* .

Lemma 2. Let V be a variety.

- (i) $U_{k,V}$ is a sheaf on V .
- (ii) $H^i(V, U_{k,V}) = H^i(V, G_{m,V})$ for all $i \geq 1$.
- (iii) $\text{Pic}(V) = H^1(V, U_{k,V})$.

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Proof. Let D denote the sheaf cokernel of $(k^*)_V \rightarrow G_{m,V}$. Let W be an open subset of V . Then $((k^*)_V)|_W = (k^*)_W$ and $(G_{m,V})|_W = G_{m,W}$, and hence there is an exact sequence

$$(+) \quad 1 \rightarrow (k^*)_W \rightarrow G_{m,W} \rightarrow (D|_W) \rightarrow 1$$

of sheaves on W . The long exact cohomology sequence begins

$$0 \rightarrow (k^*)_W \rightarrow G_{m,W}(W) \rightarrow D(W) \rightarrow H^1(W, (k^*)_W) = 0,$$

and hence $D(W) = U_{k,V}(W)$. This proves part (i). For part (ii), we use the sequence (+) for $W = V$ and $D = U_{k,V}$ and the fact that $H^i(V, (k^*)_V) = H^{i+1}(V, (k^*)_V) = 0$ for all $i \geq 1$. Part (iii) is immediate from (ii) and the fact that $\text{Pic}(V) = H^1(V, G_{m,V})$.

Definition 3. A morphism $E \rightarrow B$ of varieties is a *Zariski fibration* with fibre F if there is an open cover $\{W_i\}$ of B such that $E \times_B W_i$ is isomorphic with $F \times W_i$ (over W_i) for each i .

Lemma 4. Let $f: E \rightarrow B$ be a Zariski fibration with fibre F . Then there is an exact sequence

$$1 \rightarrow U_{k,B} \rightarrow f_* U_{k,E} \rightarrow (U_{k,F}(F))_B \rightarrow 1$$

of sheaves on B .

Proof. The first map is defined as follows: for an open set W of B , for an element a of $U_{k,B}(W)$ and for an element b of $G_{m,B}(W)$ representing a , send a to class of $b \circ f$ in $f_* U_{k,E}(W) = U_{k,E}(f^{-1}(W))$. This is trivially an injection. Let C denote its cokernel. If $\{W_i\}$ is an open cover of B as in Definition 3, then $f_* U_{k,E}(W) = U_{k,E}(F \times W)$ for every open set W contained in some W_i . By Rosenlicht's lemma [2, Lemma 3, p.7], the natural map $U_{k,F}(F) \times U_{k,B}(W) \rightarrow U_{k,E}(F \times W)$ is an isomorphism. It follows that $(f_* U_{k,E})|_{W_i} = (U_{k,B})|_{W_i} \times (U_{k,F}(F))_{W_i}$ and, hence, that $C|_{W_i} = (U_{k,F}(F))_{W_i}$. Thus C is locally constant, and it follows that $C = (U_{k,F}(F))_B$.

Theorem 5. Let $f: E \rightarrow B$ be a Zariski fibration with fibre F . Suppose that, for all sufficiently small open sets W of B the natural map $\text{Pic}(F) \times \text{Pic}(W) \rightarrow \text{Pic}(F \times W)$ is an isomorphism. Then there is an exact sequence

$$\begin{aligned} 0 \rightarrow U_{k,B}(B) \rightarrow U_{k,E}(E) \rightarrow U_{k,F}(F) \rightarrow \text{Pic}(B) \rightarrow \text{Pic}(E) \\ \rightarrow \text{Pic}(F) \rightarrow H^2(B, G_{m,B}) \rightarrow H^2(E, G_{m,E}). \end{aligned}$$

Proof. Consider the Leray spectral sequence $H^p(B, R^q f_* U_{k,E}) \Rightarrow H^n(E, U_{k,E})$. The exact sequence of low degree terms is

$$\begin{aligned} 0 \rightarrow H^1(B, f_* U_{k,E}) \rightarrow H^1(E, U_{k,E}) \rightarrow H^0(B, R^1 f_* U_{k,E}) \\ \rightarrow H^2(B, f_* U_{k,E}) \rightarrow H^2(E, U_{k,E}). \end{aligned}$$

We consider the terms separately, beginning with the third. $R^1 f_* U_{k,E}$ is the sheaf associated to the presheaf $U \rightarrow H^1(f^{-1}U, U_{k,E})$. By Lemma 2(i+), this is the presheaf $f_* \text{Pic}$. We have a map of presheaves on B , $\text{Pic} \rightarrow f_* \text{Pic}$. Let C be the presheaf cokernel. On sufficiently small open sets W of B we have $f_* \text{Pic} = \text{Pic} \times (\text{Pic}(F))_W$ as presheaves on W . The sheaf sequence associated to the exact sequence $\text{Pic} \rightarrow f_* \text{Pic} \rightarrow C \rightarrow 0$ then yields an isomorphism of $R^1 f_* U_{k,E}$ and the sheaf associated to C , since the sheaf associated to Pic is 0. But C is locally constant and locally isomorphic to $(\text{Pic}(F))_B$, hence $R^1 f_* U_{k,E} = (\text{Pic}(F))_B$, and $H^0(B, R^1 f_* U_{k,E}) = \text{Pic}(F)$. By Lemma 2(ii) and (iii) we have $H^1(E, U_{k,E}) = \text{Pic}(E)$ and $H^2(E, U_{k,E}) = H^2(E, G_{m,E})$. The long exact sequence of cohomology arising from the sheaf sequence of Lemma 4 shows that $H^2(B, f_* U_{k,E}) = H^2(B, U_{k,B})$ since $H^1(B, (U_{k,F}(F))_B) = H^2(B, (U_{k,F}(F))_B) = 0$, and hence, by Lemma 2(ii), $H^2(B, f_* U_{k,E}) = H^2(B, G_{m,B})$. The first part of that long exact sequence is

$$0 \rightarrow U_{k,B}(B) \rightarrow U_{k,E}(E) \rightarrow U_{k,F}(F) \rightarrow H^1(B, U_{k,B}) \rightarrow H^1(B, f_* U_{k,E}) \rightarrow 0.$$

By Lemma 2(iii), $H^1(B, U_{k,B}) = \text{Pic}(B)$, and now splicing this sequence to the sequence obtained from the Leray spectral sequence yields the result.

Remark. Let $f: E \rightarrow B$ be a Zariski fibration with fibre F . Suppose that for all sufficiently small open sets W of B the natural map $\text{Pic}(F) \times \text{Pic}(W) \rightarrow \text{Pic}(F \times W)$ is an isomorphism, and that F and B are smooth. Then the map $\text{Pic}(E) \rightarrow \text{Pic}(F)$ of Theorem 5 is onto.

Proof. We need to show that the kernel of $H^2(B, G_m) \rightarrow H^2(E, G_m)$ is zero. Let W be an open set of B with $f^{-1}(W) = F \times W$. We have a commutative diagram

$$\begin{array}{ccc} H^2(B, G_{m,B}) & \longrightarrow & H^2(E, G_{m,E}) \\ \downarrow & & \downarrow \\ H^2(W, G_{m,W}) & \longrightarrow & H^2(F \times W, G_{m,F \times W}) \end{array}$$

and the bottom map is clearly injective. By [1, 1.8, p. 104], the vertical maps are injective and the result follows.

The exact sequence of Theorem 5 is the generalization of the Fossum-Iverson sequence, and the above remark explains why their sequence stops with $\text{Pic}(F)$.

Fossum and Iverson show that if F and W are smooth varieties, and one is rational, then $\text{Pic}(F \times W) = \text{Pic}(F) \times \text{Pic}(W)$ [2, Corollary 6, p. 11]. This then gives some conditions for the hypotheses of Theorem 5 to obtain. (We remark that this product formula does not follow from the spectral sequence argument given here, but is a hypothesis for it.) For example, one then obtains the Fossum-Iverson sequence under the assumptions that B is rational and F smooth.

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