FINITE GROUPS WITH SEMISIMPLE ENDOMORPHISM RINGS

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ABSTRACT. If the near ring $E(G)$ generated by the endomorphisms of a finite group G is a semisimple ring, or more generally, has square-free characteristic then it is shown that G is abelian, a direct sum of elementary p -groups.

A near ring is a triple $(R, +, \cdot) \equiv R$ such that $(R, +)$ is a group, $(R, \cdot)$ is a semigroup, and $\cdot$ is left distributive over $+$. Let G be an arbitrary group, written additively though not assumed commutative. The near ring generated by the endomorphisms of G under pointwise addition and composition of maps will be denoted by $E(G)$. It is well known that when G is abelian $E(G)$ is a ring. However the converse is not true; there exist non-abelian groups G for which $E(G)$ is a ring [3].

A. J. Chandy [2] has shown that if $E(G)$ is a ring then G must be nilpotent (of degree ≤ 3). Hence if G is also finite, it will be the direct sum of its fully invariant Sylow p -subgroups, G_p (p a prime). The projections onto the G_p will decompose $E(G)$ into the direct sum of the $E(G_p)$. Obviously each $E(G_p)$ will also be a ring.

By the *characteristic* of a ring R with 1, let us mean the least positive n such that $n \cdot 1 = 0$; or 0 if no such n exists. Then the characteristic of $E(G)$ will be the exponent of G . Clearly if the characteristic n is not square-free then $k \cdot 1$ generates a nonzero nilpotent ideal of $E(G)$ where k is the product of the distinct primes dividing n . Thus, $E(G)$ will be a semisimple ring only if it has square-free characteristic.

Furthermore, if n is square-free then $nG = 0$ implies $pG_p = 0$. We can now use C. J. Maxson's result [4, sp. case Theorem 2.1 and Lemma 3.2], that if $pG_p = 0$ and $E(G_p)$ is a ring then G_p is abelian, and conclude that G is a direct sum of elementary abelian p -groups.

The above considerations establish the implications $(a) \Rightarrow (b) \Rightarrow (c)$ of the Theorem below. The final implication $(c) \Rightarrow (a)$ is immediate.

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Theorem. *For a finite group G , the following conditions are equivalent:*

- (a) *$E(G)$ is a semisimple ring,*
- (b) *$E(G)$ is a ring with square-free characteristic, and*
- (c) *G is a direct sum of elementary abelian p -groups.*

By a result of Beidleman [1, Theorem 2.2], if G is finite and solvable with $E(G)$ a semisimple near ring, then $E(G)$ is in fact a ring. Hence we get a

Corollary. *If G is a finite nonabelian solvable group, then $E(G)$ is not semisimple.*

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