APPROXIMATION OF FUNCTIONS BY CONVEXITY PRESERVING CONTINUOUS LINEAR OPERATORS

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ABSTRACT. In this note, extension of the ideas used by Korovkin, Bojanic and Roulier, Karlin and Studden, Lorentz and Schumaker are studied in order to treat various aspects of approximation of continuous functions by operators which preserve generalized convexity in the sense of Karlin and Studden, and Ziegler.

There is an intensive study of the approximation by positive linear operators. The fundamental ideas for direct estimates appear in the book of P. P. Korovkin [4]. The main ideas for the saturation problem are concentrated in DeVore's book [3]. S. Karlin and W. J. Studden [5] give a complete study of the Chebyšev systems and their connection with the concept of approximation of functions. The problem of saturation of positive operators in this general context was studied lately by G. G. Lorentz and L. L. Schumaker [6].

In this note we wish to announce that one may extend the ideas used by P. P. Korovkin [4], R. Bojanic and J. A. Roulier [2], S. Karlin and W. J. Studden [5], and G. G. Lorentz and L. L. Schumaker [6] to treat the various aspects of approximation of continuous functions by operators which preserve generalized convexity in the sense of Z. Ziegler [7] and S. Karlin and W. J. Studden [5].

Let $w_0, \dots, w_{l+2}$ ($l \geq 0$) be a finite sequence of continuous strictly positive functions in an open interval (a, b) with $w_j \in C^{l+2-j}(a, b)$, $j = 0, \dots, l+2$. Let $D_0 f(x), \dots, D_{l+2} f(x)$ be a sequence of operators defined by

$$(1) \quad D_j f(x) = [f(x)/w_j(x)]';$$

also let D^j be the operators

$$(2) \quad D^j = D_j D_{j-1} \cdots D_0, \quad j = 0, 1, \dots, l+2,$$

and

$$(3) \quad \bar{D}^j = D^{j-1}/w_j, \quad j = 1, \dots, l+2.$$

Define the functions

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$$\begin{aligned}
v_0(x) &= w_0(x), \\
v_1(x) &= w_0(x) \int_a^x w_1(\zeta_1) d\zeta_1, \\
&\vdots \\
v_{l+2}(x) &= w_0(x) \int_a^x w_1(\zeta_1) d\zeta_1 \int_a^{\zeta_1} w_2(\zeta_2) d\zeta_2 \cdots \int_a^{\zeta_{l+1}} w_{l+2}(\zeta_{l+2}) d\zeta_{l+2}.
\end{aligned}$$

It is known that $v_0, \dots, v_{j-1}$ form a fundamental set of solutions of the differential equation $\bar{D}_j v = 0$ and $\bar{D}_j v_j = 1$. $v_0, \dots, v_{l+2}$ also form an extended complete Chebyšev system and $v_k \in C^{l+2}(a, b)$, $k = 0, \dots, l+2$.

A function $f(x)$ on $[a, b]$ is convex with respect to $v_0, \dots, v_{l+1}$ ($l \geq 0$) if

$$(5) \quad U \left(\begin{matrix} v_0, \dots, v_{l+1}, f \\ x_0, \dots, x_{l+1}, x_{l+2} \end{matrix} \right) \equiv \begin{vmatrix} v_0(x_0) & v_0(x_{l+2}) \\ \vdots & \vdots \\ v_{l+1}(x_0) & v_{l+1}(x_{l+2}) \\ f(x_0) & f(x_{l+2}) \end{vmatrix} \geq 0$$

for any $a \leq x_0 < x_1 < \dots < x_{l+2} \leq b$, and we shall write $f \in C(v_0, \dots, v_{l+1})$.

For any function $f(x)$ on (a, b) and any set of points $a \leq x_0 < \dots < x_{l+1} \leq b$, there exists a unique interpolatory linear function $\phi = \sum_{j=0}^{l+1} a_j v_j$ s.t. $f(x_j) = \phi(x_j)$ for $j = 0, 1, \dots, l+1$.

The following lemmas will be used in the sequel.

Lemma 1. *If ϕ is linear and interpolates a continuous function $f \in C(v_0, \dots, v_{l+1})$ at $a \leq x_1 < \dots < x_{l+2} \leq b$, then $\bar{D}^l f \leq \bar{D}^l \phi$ in some interval (z_1, z_2) with $x_1 < z_1 < z_2 < x_{l+2}$.*

The proof follows using Theorem 3.1 of [6] and a generalized form of Rolle's theorem.

Lemma 2. *If $f \in C(v_0, \dots, v_{l+1})$, and ϕ is linear and satisfies $\bar{D}^j \phi(x_0) = \bar{D}^j f(x_0)$, $j = 0, \dots, l+1$, $a < x_0 < b$, then $\bar{D}^j f(x) \geq \bar{D}^j \phi(x)$ for $a \leq x \leq b$.*

Use of Lemma 2.2 in [6] and Theorem 3.1 in [5] lead to the proof.

Using these lemmas we can start the study of operators which preserve convexity.

It is possible to reformulate some results of Z. Ziegler [7] and S. Karlin and W. J. Studden [5] in order to obtain a characterization of such operators (see also R. Bojanic and J. A. Roulier [2]).

Theorem 1. *Let $T: C[a, b] \rightarrow C[a, b]$ be a continuous linear operator.*

In order that $T: C(v_0, \dots, v_{l+1}) \rightarrow C(v_0, \dots, v_{l+1})$ ($l \geq 0$) (i.e. a convexity preserving positive (c.p.c.l.) operator), it is necessary and sufficient that

- (i) for every linear function ϕ , $T(\phi, x)$ is also a linear function,
- (ii) $T(\phi_{l+1}(t, c), x) \in C(v_0, \dots, v_{l+2})$ for every $c \in (a, b)$, where

$$(6) \quad \phi_{l+1}(t, c) = \begin{cases} w_0(t) \int_x^t w_1(\zeta_1) \cdots \int_x^{\zeta_l} w_{l+1}(\zeta_{l+1}) d\zeta_{l+1} \cdots d\zeta_1, & x \leq t \leq b, \\ 0, & a \leq t < x, \end{cases}$$

Using a procedure similar to that of P. P. Korovkin [4], it is possible to find a criterion to characterize those sequences of c.p.c.l. operators which converge strongly to the identity and also to find asymptotic relations of Voronskaja's type.

Theorem 2. Let T_n ($n \geq 1$) be a sequence of c.p.c.l. operators and suppose that $\|T_n\| \leq M < \infty$. T_n converges strongly to the identity iff

$$(7) \quad \lim_{n \rightarrow \infty} T_n v_j = v_j, \quad j = 0, 1, \dots, l+2.$$

Theorem 3. If $f \in C^{(l+2)}(a, b)$ and T_n ($n \geq 1$) is a sequence of c.p.c.l. operators which converges strongly to the identity, then

$$(8) \quad \lim_{n \rightarrow \infty} \frac{T_n(f, x) - f(x)}{T_n(v_{l+2}, x)} = \bar{D}^{l+2}f(x),$$

and

$$(9) \quad \lim_{n \rightarrow \infty} \frac{\bar{D}^l T_n(f, x) - \bar{D}^l f(x)}{\bar{D}^l T_n(v_{l+2}, x)} = \bar{D}^{l+2}f(x).$$

Then when we study the saturation problem it is natural to assume that T_n satisfy an asymptotic formula of the following type:

$$(10) \quad \lim_{n \rightarrow \infty} \lambda_n \bar{D}^l r_n(f, x) = \bar{D}^l \rho(x) Df(x), \quad a < x < b,$$

where $\lambda_n > 0$ converges to $+\infty$, $r_n(f, x) \equiv T_n(f, x) - f(x)$, $\bar{D}^l \rho(x) \geq 0$ on $[a, b]$ and $\bar{D}^l \rho(x) > 0$ on (a, b) , and D is a differential operator of type $\bar{D}^k$ with $k \geq l+2$. The following result is an extension of [6, Lemma 4.1].

Theorem 4. If T_n ($n \geq 0$) is a sequence of c.p.c.l. operators which satisfies

$$(11) \quad \lambda_n \bar{D}^l r_n(f, x) \geq -M \bar{D}^l \rho(x) + o(1)$$

with $M \geq 0$, then

$$(12) \quad f + M v_{l+2} \in C(v_0, \dots, v_{l+1}).$$

In the proof of this theorem, Lemma 1 is needed.

Corollary 1. *A sequence of c.p.c.l. operators can satisfy a relation of type (10) only if $k = l + 2$. This follows from Theorem 4, choosing $f = -v_{l+2}$ and $M = 0$.*

Now we state the inverse result to Theorem 4.

Theorem 5. *If T_n ($n \geq 1$) is a sequence of c.p.c.l. operators and $f + Mv_{l+2} \in C(v_0, \dots, v_{l+1})$, then*

$$(13) \quad \lambda_n \bar{D}^l r_n(f, x) > -M \bar{D}^l \rho(x) + o(1).$$

This theorem is a simple consequence of Lemma 2.

Combining Theorems 4, 5 and results from [5, §3] we find

Corollary 2. *A function $f \in C^l[a, b]$ satisfies*

$$(14) \quad \lambda_n |\bar{D}^l r_n(f, x)| \leq M \bar{D}^l \rho(x) + o(1), \quad a < x < b,$$

iff $f \in \text{Lip}_M^1$ with respect to $v_0, v_1, \dots, v_{l+2}$ (i.e. $\bar{D}^{l+1}f(x)$ is absolutely continuous and $|\bar{D}^{l+2}f(x)| \leq M$ [6]).

Corollary 3. *If $M = 0$, then if f satisfies a relation of the type $\lambda_n \bar{D}^l r_n(f, x) = o(1)$ for $a < x < b$, then f is linear. This generalizes a result of Bajšanski and Bojanic [1] and also a remark of [6].*

Corollary 4. *If $T_n v_j = v_j$, $j = 0, 1, \dots, l + 1$, we can remove the $o(1)$ from Corollary 2.*

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