BOUNDED MULTIPLIER CONVERGENCE IN MEASURE OF RANDOM VECTOR SERIES

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ABSTRACT. If the series $\sum f_i$ of random vectors with values in a Banach space converges unconditionally in measure, then, for each $(\lambda_i) \in l^\infty$, the series $\sum \lambda_i f_i$ also converges unconditionally in measure.

Recall that the series of elements of a topological group is said to be unconditionally convergent if it remains convergent after arbitrary permutation of its terms. The following proposition is, essentially, a classical result due to Orlicz (cf. also [2, Theorem 1]).

Proposition 1. *Let (x_i) be a sequence of elements of a sequentially complete topological abelian group. Then the series $\sum x_i$ converges unconditionally if and only if for every sequence of $\epsilon_i = \pm 1$ the series $\sum \epsilon_i x_i$ converges.*

Now, let $(\mathcal{X}, \|\cdot\|)$ be a Banach space and let $(T, \mathcal{A}, \mu)$ be a probability system. By definition $L^0(T, \mathcal{A}, \mu; \mathcal{X}) = L^0(\mathcal{X})$ is the linear complete metric space of all measurable functions on $(T, \mathcal{A}, \mu)$ with values in $\mathcal{X}$ (i.e. μ -a.e. limits of simple functions) equipped with the quasi-norm

$$\|f\|_0 = \min \{c : c^{-1}\mu\{t : \|f(t)\| > c\} \leq 1\}.$$

Theorem 1. *If for all choices of $\epsilon_i = \pm 1$ the series $\sum \epsilon_i f_i$, $f_i \in L^0(\mathcal{X})$, converges in $L^0(\mathcal{X})$ then also for each sequence $(\lambda_i) \in l^\infty$ the series $\sum \lambda_i f_i$ converges in $L^0(\mathcal{X})$.*

In the special case $\mathcal{X} = \mathbb{R}$ this theorem was obtained by Maurey and Pisier [1] but our method of proof is much shorter and more elementary. Proposition 1 and Theorem 1 give us immediately the following theorem which confirms a long-standing conjecture (cf. [3, Problem III.6.8]).

Theorem 2. *In $L^0(\mathcal{X})$, unconditional convergence of the series $\sum f_i$ is equivalent to convergence of all the series $\sum \lambda_i f_i$, $(\lambda_i) \in l^\infty$.*

Notice that Theorem 2 may be used as a tool in constructing an integral of any bounded real measurable function with respect to the bounded (random) measure with values in $L^0(\mathcal{X})$ (cf. [3, Theorem III.6.2]). For integrals with

Received by the editors July 11, 1974.

AMS (MOS) subject classifications (1970). Primary 46E40; Secondary 46G10.

Key words and phrases. Unconditional convergence, bounded multiplier, random vector.

respect to random measures with independent values on disjoint sets more information is available from [4].

Proof of Theorem 1. By [2, Lemma 2], given the sequence $\lambda = (\lambda_i)$, $|\lambda_i| \leq 1$, for each sequence $(x_i) \in \mathcal{X}$ and arbitrary $n \in \mathbb{N}$, we have the inequality

$$P \left\{ (\epsilon_i) : \left\| \sum_{i=1}^n \epsilon_i x_i \right\| > \frac{1}{8} \left\| \sum_{i=1}^n \lambda_i x_i \right\| \right\} \geq \frac{1}{8},$$

where $P = (P_0 + P_\lambda)/2$, and P_λ is the probability on $\{-1, +1\}^{\mathbb{N}}$ such that the random variables $\{-1, +1\}^{\mathbb{N}} \ni (\epsilon_i) \rightarrow \epsilon_j$, $j = 1, 2, \dots$, are independent and $\int \epsilon_j dP_\lambda = \lambda_j$. Now, given the sequence $(f_i) \subset L^0(\mathcal{X})$, for each $t \in T$, we have that

$$P \left\{ (\epsilon_i) : \left\| \sum_{i=1}^n \epsilon_i f_i(t) \right\| > \frac{1}{8} \left\| \sum_{i=1}^n \lambda_i f_i(t) \right\| \right\} \geq \frac{1}{8}, \quad n \in \mathbb{N}.$$

Integrating this inequality with respect to the measure μ we get that

$$(\mu \times P) \left\{ (t, (\epsilon_i)) : \left\| \sum_{i=1}^n \epsilon_i f_i(t) \right\| > \frac{1}{8} \left\| \sum_{i=1}^n \lambda_i f_i(t) \right\| \right\} \geq \frac{1}{8}, \quad n \in \mathbb{N}.$$

By the Fubini theorem (notice that $\max$ is taken over the finite set)

$$(1) \quad \max_{(\epsilon_i)} \mu \left\{ t : \left\| \sum_{i=1}^n \epsilon_i f_i(t) \right\| > \frac{1}{8} \left\| \sum_{i=1}^n \lambda_i f_i(t) \right\| \right\} \geq \frac{1}{8}, \quad n \in \mathbb{N}.$$

Now, let

$$T_1 \stackrel{\text{df}}{=} \left\{ t : \frac{1}{8} \left\| \sum_{i=1}^n \lambda_i f_i(t) \right\| > c \right\}, \quad c > 0,$$

and $\mu(T_1) > 0$. Put $\mu_1(A) \stackrel{\text{df}}{=} \mu(A \cap T_1)/\mu(T_1)$, $A \in \mathcal{A}$. Applying (1) to the measure μ_1 (instead of μ) we have that

$$\begin{aligned} & \max_{(\epsilon_i)} \mu_1 \left\{ t : \left\| \sum_{i=1}^n \epsilon_i f_i(t) \right\| > c \right\} \\ & \geq \max_{(\epsilon_i)} \mu_1 \left\{ t : \left\| \sum_{i=1}^n \epsilon_i f_i(t) \right\| > \frac{1}{8} \left\| \sum_{i=1}^n \lambda_i f_i(t) \right\| \right\} \geq \frac{1}{8}, \quad n \in \mathbb{N}, \end{aligned}$$

and, eventually, by definition of μ_1 , we get that

$$(2) \quad \max_{(\epsilon_i)} \mu \left\{ t : \left\| \sum_{i=1}^n \epsilon_i f_i(t) \right\| > c \right\} \geq \frac{1}{8} \mu \left\{ t : \left\| \sum_{i=1}^n \lambda_i f_i(t) \right\| > 8c \right\}, \quad n \in \mathbb{N},$$

which concludes the proof of Theorem 1.

Notice that inequality (2) implies that for each $c > 0$

$$\left\{ c : \frac{1}{c} \max_{(\epsilon_i)} \mu \left\{ t : \left\| \sum_{i=1}^n \epsilon_i f_i(t) \right\| > c \right\} \leq 1 \right\} \\ \subset \left\{ \frac{1}{3} c : \frac{1}{c} \mu \left\{ t : \left\| \sum_{i=1}^n \lambda_i f_i(t) \right\| > c \right\} \leq 1 \right\},$$

and that the above inclusion and the definition of the quasi-norm $\|\cdot\|_0$ give us also the following quantitative result.

Theorem 3. Let $(f_i) \subset L^0(\mathcal{X})$ and $(\lambda_i) \in l^\infty$, $|\lambda_i| \leq 1$. Then for each $n \in \mathbb{N}$

$$\left\| \sum_{i=1}^n \lambda_i f_i \right\|_0 \leq 3 \max_{(\epsilon_i)} \left\| \sum_{i=1}^n \epsilon_i f_i \right\|_0.$$

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