ON THE NUMBER OF GENERATORS OF POWERS OF AN IDEAL

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ABSTRACT. Let I be an ideal of a quasi-local ring. In this note we consider the question of how small—in terms of numbers of generators—the powers of the ideal I can be.

In this note all rings are commutative with identity. If I is an ideal of a ring, $\nu(I)$ denotes the minimal number, which may be infinite, of generators of I .

Proposition 1. *Let I be an ideal of a quasi-local ring A . If, for some integer $t \geq 1$, $\nu(I^t) = 1$, then either $\nu(I^k) = 1$ for all positive integers k or I consists of zero divisors. If, for some integer $t > 1$, $\nu(I^t) = 2$, then either $\nu(I^k) = 2$ for all positive integers k or I consists of zero divisors.*

We will use the following lemmas in the proof of the proposition.

Lemma 2. *Let I be an ideal of a ring R . Let J and K be subsets of I such that $J \subseteq I$ and $K \subseteq I^{t-1}$ for some positive integer t . If $I^t = JKR$, then $I^{t+1} = J^2KR$.*

Proof. $I^{t+1} = IJKR = JIKR \subseteq JI^t = J^2KR$.

It is an immediate consequence of Lemma 2 that if some power I^t of an ideal I is finitely generated then so is every higher power. Since I^t is generated by sums of monomials of degree t in elements of I , take J to be the ideal generated by all the elements of I which appear in the monomials in such a generating set. Then J is a finitely generated ideal contained in I , $I^t = J^t$ and, by Lemma 2, $I^{t+k} = J^{t+k}$ for $k \geq 0$.

If an element x in I appears in j monomials in some minimal generating set for I^t we will say that I^t has an *overlap of length j* . It follows from Lemma 2 that if $\nu(I^t) = n$ and if I^t has an overlap of length n , then $\nu(I^{t+k}) \leq n$ for $k \geq 0$. (In particular, if an ideal I in a quasi-local ring has some power principal, then every higher power is principal.)

If $(A, \mathfrak{m})$ is a quasi-local ring and u an indeterminate we will often make use of the faithfully flat change of rings $A \rightarrow A(u) = A[u]_{\mathfrak{m}A[u]}$.

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Lemma 3. *Let $(A, \mathfrak{m})$ be a quasi-local ring and I a finitely generated ideal. Let t be any integer > 1 . Then either I^t is principal or $I^t A(u)$ has an overlap of length 2.*

Proof. Let $I = (x_1, \dots, x_n)$ and suppose that $v(I^t) > 1$, say

$$I^t = (x_{i_1}^{\alpha_1} \dots x_{i_r}^{\alpha_r}, \dots, x_{j_1}^{\beta_1} \dots x_{j_s}^{\beta_s})$$

with α_i, β_j positive integers such that $\alpha_1 + \dots + \alpha_r = \beta_1 + \dots + \beta_s = t$. Assume that every x_i appears in at most one monomial in this generating set for I^t . Now

$$(*) \quad x_{i_1}^{\beta_1-1} x_{j_1}^{\beta_2} \dots x_{j_s}^{\beta_s} = ax_{i_1}^{\alpha_1} \dots x_{i_r}^{\alpha_r} + \dots + bx_{j_1}^{\beta_1} \dots x_{j_s}^{\beta_s}$$

for some $a, \dots, b \in A$. If one of the coefficients $a, \dots, b$ is not in $\mathfrak{m}$ we can replace the generator it multiplies by $x_{i_1}^{\beta_1-1} x_{j_1}^{\beta_2} \dots x_{j_s}^{\beta_s}$ and produce an overlap in I^t itself. So we assume that all the coefficients are in $\mathfrak{m}$.

We pass to the ring $A(u)$ and change the basis of $IA(u)$ as follows:

$$x_{i_1} = y_{i_1} + uy_{j_i}; \quad x_k = y_k \quad \text{for } k \neq i_1.$$

Then

$$IA(u) = (y_1, \dots, y_n) \quad \text{and} \quad I^t A(u) = (y_{i_1}^{\alpha_1} \dots y_{i_r}^{\alpha_r}, \dots, y_{j_1}^{\beta_1} \dots y_{j_s}^{\beta_s}) A(u),$$

for

$$I^t \subseteq (y_{i_1}^{\alpha_1} \dots y_{i_r}^{\alpha_r}, \dots, y_{j_1}^{\beta_1} \dots y_{j_s}^{\beta_s}) A[u] + u I^t A[u].$$

Now (*) gives

$$y_{i_1} y_{j_1}^{\beta_1-1} y_{j_2}^{\beta_2} \dots y_{j_s}^{\beta_s} + (u - b) y_{j_1}^{\beta_1} \dots y_{j_s}^{\beta_s} \in \mathfrak{m} I^t A(u).$$

Hence, if we substitute $y_{i_1} y_{j_1}^{\beta_1-1} y_{j_2}^{\beta_2} \dots y_{j_s}^{\beta_s}$ for the generator $y_{j_1}^{\beta_1} \dots y_{j_s}^{\beta_s}$ of $I^t A(u)$, we have that $I^t A(u)$ has an overlap of length 2.

Corollary. *Let A be a quasi-local ring and I an ideal such that for some $t > 1$, $v(I^t) = 2$. Then $v(I^{t+k}) \leq 2$ for all $k \geq 0$.*

Proof. By Lemma 2 we may assume that I is finitely generated. $I^t A(u)$ has 2 generators and, by Lemma 3, an overlap of length 2. Hence $v(I^{t+k} A(u)) \leq 2$ and $v(I^{t+k}) \leq 2$.

Proof of Proposition 1. If I is not principal, let $t > 1$ be the least integer such that $v(I^t) = 1$. Say $I^t = (x_1^{\alpha_1} \dots x_r^{\alpha_r})$, where the α_i are positive integers such that $\alpha_1 + \dots + \alpha_r = t$ and the x_i are in I . Since it is enough to show that some x_i is a zero divisor we may assume that $I = (x_1, \dots, x_r)$ and that $x_1^{\alpha_1-1} x_2^{\alpha_2} \dots x_r^{\alpha_r}$ is one of the generators in a minimal basis for I^{t-1} . Let z be another such generator. Then the relation $x_1 z = bx_1^{\alpha_1} \dots$

$x_r^{\alpha_r}$, for some b in A , shows that x_1 is a zero divisor. To prove the second statement, suppose that $t > 1$ is a positive integer such that $v(I^t) = 2$ and $v(I^{t-1}) > 2$. By passing to the ring $A(u)$ we may assume that I^t has a minimal generating set of the form $x_1 x_2^{\alpha_2} \cdots x_r^{\alpha_r}, x_1 x_{r+1}^{\alpha_{r+1}} \cdots x_s^{\alpha_s}$ with $x_1, \dots, x_s \in I$ and α_i positive integers such that $\alpha_1 + \cdots + \alpha_r = \alpha_{r+1} + \cdots + \alpha_s = t - 1$. Then there is a z in $I^{t-1} \setminus (x_2^{\alpha_2} \cdots x_r^{\alpha_r}, x_{r+1}^{\alpha_{r+1}} \cdots x_s^{\alpha_s})$. The relation

$$zx_1 = ax_1 x_2^{\alpha_2} \cdots x_r^{\alpha_r} + bx_1 x_{r+1}^{\alpha_{r+1}} \cdots x_s^{\alpha_s},$$

for some a, b in A , shows that x_1 is a zero divisor and that $IA(u)$ consists of zero divisors. Hence (cf. [3]) I consists of zero divisors.

The following example was shown to me by P. Eakin and W. Heinzer. (The reference [1] was supplied by the referee.) The example demonstrates that "very big" ideals in quasi-local domains can have powers generated by 3 elements. Let $n \in \{1, 2, \dots, \infty\}$. Let S be any ring having an ideal J minimally generated by n elements. Let $R = S[[x^4, x^5, Jx^{11}]]$, where x is an indeterminate, and consider the ideal $I = (x^4, x^5, Jx^{11})R$. $v(I) = n + 2$, but $v(I^2) = 3$ since $I^2 = (x^8, x^9, x^{10})R$. Variation of the semigroup used as exponents for x leads to many such examples. Notice, however, that the domain R has dimension greater than one. What happens in the one-dimensional local case, except for some very special ideals (e.g. [2], [4]), is an open question. It is possible that the following generalization of Lemma 3 holds: if I is an ideal of a quasi-local ring A with $v(I^t) = k$ where $t \geq k$, then $v(I^{t+j}) \leq k$ for all $j \geq 0$.^{*} The combinatorial methods used above give a rather lengthy verification of this for $k = 3$, but this does not seem to be a good way to attack the general case.

^{*} Added in proof. This conjecture has been proved by Eakin and Sathaye, *Prestable ideals*, J. Algebra (to appear). Also pertinent are the papers *A note on the Hilbert function of a one-dimensional Cohen-Macaulay ring* (preprint) by Herzog and Waldi, and *Hilbert functions of graded algebras* (preprint) by Stanley.

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