ON THE CORADICAL OF A FINITE-DIMENSIONAL HOPF ALGEBRA

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ABSTRACT. Examples are constructed showing that the coradical of a finite-dimensional Hopf algebra over any algebraically closed field is not necessarily a subalgebra (hence the Jacobson radical is not a Hopf ideal in general). The square of the antipode may induce a permutation on the simple subcoalgebras of dimension > 1 of arbitrarily high order.

0. Introduction. In this paper we discuss the coradical (and thus naturally the Jacobson radical) of a finite-dimensional Hopf algebra over a field k . Over any algebraically closed field finite-dimensional examples are constructed such that the coradical is not a sub-Hopf algebra (hence the Jacobson radical is not necessarily a Hopf ideal). In [2] it is shown that the square of the antipode maps each simple subcoalgebra into itself in the cosemisimple case. In general this does not happen; there exist finite-dimensional Hopf algebras over any algebraically closed field k with antipode s of order $2n$ (n large enough) such that $t = s^2$ induces a permutation on the simple subcoalgebras of dimension > 1 or order n .

The Hopf algebras in this paper arise surprisingly enough as dual Hopf algebras of pointed ones which have a relatively simple description. The basic ideas of the constructions are taken from [5] and [3]. The moderate amount of notation and results concerning Hopf algebras we assume can be found in [4], and the algebra we take for granted is adequately covered in [1]. All vector spaces, coalgebras, Hopf algebras, etc. will be over a field k .

1. Preliminaries. The examples we construct in this paper will be dual Hopf algebras of finite-dimensional pointed Hopf algebras. The basic building block is Taft's example of [5] as described below.

1.1. Let k be a field containing a primitive n th root of unity ω , $n > 1$. The free algebra $\mathcal{H} = k\{A, Z\}$ on noncommuting indeterminants A and Z is a bialgebra where $\Delta: \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ is determined by

$$\Delta A = A \otimes A \quad \text{and} \quad \Delta Z = Z \otimes A + 1 \otimes Z,$$

Received by the editors September 3, 1974.

AMS (MOS) subject classifications (1970). Primary 16A24; Secondary 16A21, 16A64.

Key words and phrases. Hopf algebra, Jacobson radical, order of antipode on coradical.

¹ Research done while the author was a visitor to the Institute for Advanced Study, Princeton, N. J., and supported by NSF Grant GP-42922.

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and $\epsilon: \mathcal{H} \rightarrow k$ is determined by

$$\epsilon(A) = 1 \quad \text{and} \quad \epsilon(Z) = 0.$$

The ideal γ of $\mathcal{H}$ generated by $A^n - 1$, $ZA - \omega AZ$, and Z^n is a bi-ideal, and $T_n = \mathcal{H}/\gamma$ is a Hopf algebra with antipode of order $2n$. T_n has for a basis (denoting cosets by small case letters) the set $\{a^i z^j: 0 \leq i, j < n\}$. Thus $\dim T_n = n^2$.

We will maintain the convention of denoting cosets by small case letters in §§2, 3.

Taft's example T_n and its dual T_n^* are isomorphic as Hopf algebras. For let $\alpha \in \text{Alg}(T_n, k) = G(T_n^*)$ and $\mathcal{Z} \in T_n^*$ be determined by

$$\alpha(h) = \begin{cases} \omega: h = a, \\ 0: h = z, \end{cases} \quad \text{and} \quad \mathcal{Z}(h) = \begin{cases} 1: h = a^l z, \\ 0: h = a^l z^p, p \neq 1. \end{cases}$$

One can check that the map $T_n \rightarrow T_n^*$ determined by $a \mapsto \alpha$ and $z \mapsto \mathcal{Z}$ is an isomorphism of Hopf algebras.

The following observations will be useful in 1.3 and §3.

1.2. Let A be an algebra over k , and $a, b \in A$ satisfy $ab = \omega ba$ for some $0 \neq \omega \in k$. Then:

(a) $(ab)^n = \omega^{-(n-1)n/2} a^n b^n$ for $n \geq 1$.

(b) If ω is a primitive n th root of unity then $(a + b)^n = a^n + b^n$.

(a) follows easily by induction, and (b) is essentially proven in [5].

In order to analyse the structure of the Hopf algebras in §§2 and 3 it will be necessary to show that certain algebras are semisimple. The next proposition provides sufficient criteria for our purposes.

1.3. **Proposition.** *Let B be an algebra over a field k .*

(a) *Suppose B is generated by a and z which satisfy $a^n = z^n = 1$ and $za = \omega az$, where $\omega \in k$ is a primitive n th root of unity, $n > 1$. Then $B \simeq M(n, k)$.*

(b) *Suppose k is algebraically closed, and B is generated by a, x, y which satisfy $a^n = x^n = y^n = 1$, $xa = \omega ax$, $ya = \eta ay$, and $xy = \eta^{-1}yx$, where $\omega, \eta \in k$ are primitive n th roots of unity. Then there is a $\xi \in k$ and $q > 0$ such that $(\xi a^{n-q} x^q)^n = 1$, and any maximal ideal is of the form $\mathfrak{M}_l = (y - \xi \omega^l a^{n-q} x^q)$ for some $0 \leq l < n$. Furthermore $B \simeq M(n, k) \times \cdots \times M(n, k)$ (p -times) where $p \leq n$.*

Proof. (a) Let V be a vector space over k with basis $v_0, \dots, v_{n-1}$, and define $\mathfrak{A}, \mathfrak{Z} \in \text{End}(V)$ by $\mathfrak{A}(v_i) = v_{i+1}$ (adding subscripts modulo n) and $\mathfrak{Z}(v_i) = \omega^{i-1} v_i$. Then $\text{End}(V)$ with $\mathfrak{A}$ and $\mathfrak{Z}$ satisfies the hypothesis of 1.3(a).

Let $\mathfrak{U} = k\{A, Z\}$ be the free algebra on noncommuting indeterminants A

and Z , and let $\text{End}(V) \leftarrow \mathfrak{U} \rightarrow B$ denote the obvious surjections. Let γ be the ideal of $\mathfrak{U}$ generated by $A^n - 1$, $Z^n - 1$, and $ZA - \omega AZ$. Then $\dim \mathfrak{U}/\gamma \leq n^2$, so we have factorizations $\text{End}(V) \simeq \mathfrak{U}/\gamma \rightarrow B$. This proves (a).

(b) $\eta = \omega^q$ for some $0 < q < n$. Set $\mathfrak{Y} = a^{n-q}x^q$. Then one can check that $y\mathfrak{Y} = \mathfrak{Y}y$, $x\mathfrak{Y} = \eta^{-1}\mathfrak{Y}x$, and $\mathfrak{Y}a = \eta a\mathfrak{Y}$. Since $a^{n-q}x^q = \omega^{q^2}x^qa^{n-q}$ by 1.2(a), we compute that $\mathfrak{Y}^n = \omega^{-(n-1)nq^2/2}1$. Fix $\xi \in k$ satisfying $\xi^n = \omega^{(n-1)nq^2/2}$, and consider the ideal $\mathfrak{M}_l = (y - \xi\omega^la^{n-q}x^q)$. We will show that $\mathfrak{M}_l$ is maximal (or B), and if maximal then $B/\mathfrak{M}_l \simeq M(n, k)$.

It is clear that $\mathfrak{M}_l$ is generated as a left ideal by $y - \xi\omega^la^{n-q}x^q$, and that $\mathfrak{M}_l + \mathfrak{M}_{l'} = B$ for $0 \leq l, l' < n$ distinct. Thus $\bigcap_{l=0}^{n-1} \mathfrak{M}_l = \mathfrak{M}_0 \cdot \dots \cdot \mathfrak{M}_{n-1}$. But the latter is generated by

$$(y - \xi a^{n-q}x^q)(y - \omega\xi a^{n-q}x^q) \cdot \dots \cdot (y - \omega^{n-1}\xi a^{n-q}x^q) = y^n - (\xi a^{n-q}x^q)^n = 0.$$

Thus we can conclude that $B \simeq B/\mathfrak{M}_0 \times \dots \times B/\mathfrak{M}_{n-1}$. If $\mathfrak{M}_l$ is proper, then part (a) implies that $B/\mathfrak{M}_l \simeq M(n, k)$. This completes the proof of (b). Q.E.D.

Remark. In 1.3 if $a^n = 1$, etc. is replaced by $a^n = \alpha 1$, where $0 \neq \alpha \in k$, we can conclude that B is semisimple. For if $\bar{k}$ is an algebraic closure of k , then 1.3 guarantees that $B \otimes \bar{k}$ is semisimple.

2. The order of s on the coradical. The examples in this section will show that the order of the antipode s on the coradical can be $2n$ for $n \geq 1$. In every case the Hopf algebra is generated as an algebra by the coradical, and since they are not cosemisimple, the coradical is not a sub-Hopf algebra. The coradical can have an arbitrary number of copies on $n \times n$ matrix duals.

Let H_p denote the dual Hopf algebra of the following:

2.1. Let $n > 1$ be fixed, and assume k contains a primitive n th root of unity ω . Let $\mathcal{H} = k\{A, Z\}$ be as in 1.1. Then, for given $p > 1$, the ideal γ of $\mathcal{H}$ generated by $A^{pn} - 1$, $ZA - \omega AZ$, and $Z^n - (A^n - 1)$ is a bi-ideal. $A_p = \mathcal{H}/\gamma$ is a Hopf algebra with antipode Δ determined by $\Delta(z) = -za^{-1}$, and $\{a^i z^j : 0 \leq i < pn, 0 \leq j < n\}$ is a basis for A_p . Thus $\dim H_p = \dim A_p = pn^2$.

One can verify directly the dimension statement in 2.1 by determining the rule for multiplication.

Set $g(X) = X^{p-1} + X^{p-2} + \dots + 1$. Then $X^p - 1 = (X - 1)g(X)$. Notice that $b = g(a^n)$ is in the center of A_p . Furthermore:

2.2. Lemma. Let $\mathfrak{M} = (b)$ and $B = A_p/\mathfrak{M}$. Then

(a) $\dim B = (p - 1)n^2$.

Suppose p is prime to $\text{char } k$. Then

(b) B is semisimple, and

(c) if k is algebraically closed and $\rho \in k$ is a primitive p th root of unity, then $\mathfrak{M}_{\rho^l} = (a^n - \rho^l 1)$ is maximal for $1 \leq l \leq p-1$ and $B \simeq A_p / \mathfrak{M}_{\rho} \times \dots \times A_p / \mathfrak{M}_{\rho^{p-1}}$.

Proof. (a) Since $a^n b = b$ we see that $\{b, ab, \dots, a^{n-1}b\}$ spans the set of products of the form $a^i b$. The former set is easily seen to be independent, and since b is in the center of A_p , we see $\dim \mathfrak{M} = n^2$. Thus $\dim B = \text{codim } \mathfrak{M} = (p-1)n^2$.

(b) Let $\bar{k}$ be an algebraic closure of k . If $B \otimes \bar{k}$ is semisimple, then B must be also, so (b) will follow from (c).

(c) If k is algebraically closed then $g(X) = (X - \rho) \cdot \dots \cdot (X - \rho^{p-1})$. Now $g(a^n) = 0$. Now for each l we have that $a^n - \rho^l 1$ is in the center of A_p . Thus $\mathfrak{M} = \bigcap \mathfrak{M}_{\rho^l}$ and $B \simeq A_p / \mathfrak{M}_{\rho} \times \dots \times A_p / \mathfrak{M}_{\rho^{p-1}}$ follow easily. It remains to show that $A_p / \mathfrak{M}_{\rho^l} = B_l$ is simple for $1 \leq l < p$. Notice $\dim B_l \leq n^2$, so necessarily $\dim B_l = n^2$ by (a). Thus $B_l \simeq M(n, k)$ by 1.3(a). Q.E.D.

Assume k is algebraically closed and that p is prime to $\text{char } k$. If $\eta^p = 1$ set $C_{\eta} = \mathfrak{M}_{\eta}^{\perp} = (a^n - \eta 1)^{\perp}$. Then C_{η} is a simple subcoalgebra of H_p (in fact $C_{\eta}^* \simeq A_p / \mathfrak{M}_{\eta}$ is $n \times n$ matrices) if $\eta \neq 1$; C_1 is the Hopf algebra of 1.1 since $C_1^* \simeq A_p / (a^n - 1) = T_n$ and the Hopf algebra T_n is self-dual. The following proposition sheds light on the structure of H_p .

2.3. Proposition. Suppose k is an algebraically closed field which contains a primitive n th root of unity, and let H_p be as above. Assume that p is prime to $\text{char } k$, and let $\rho \in k$ be a primitive p th root of unity. Then

$$(a) H_p = C_1 \oplus C_{\rho} \oplus \dots \oplus C_{\rho^{p-1}},$$

$$(b) C_{\mu} C_{\eta} \subseteq C_{\mu \eta} \text{ for all } p\text{th roots of unity } \mu, \eta.$$

Let s be the antipode of H_p . Then

$$(c) s \text{ has order } 2n;$$

$$(d) s(C_{\eta}) \subseteq C_{\eta^{-1}}, \text{ and } s \text{ has order } 2n \text{ on } C_{\rho}.$$

Proof. (a) Since $\bigcap_{l=0}^{p-1} \mathfrak{M}_{\rho^l} = (b(a^n - 1)) = (a^{pn} - 1) = (0)$, we have (a) by duality.

(b) $\Delta(a^n - \mu \eta 1) = (a^n - \mu 1) \otimes a^n + \mu 1 \otimes (a^n - \eta 1)$, so $\Delta(a^n - \mu \eta 1) \subseteq \mathfrak{M}_{\mu} \otimes A_p + A_p \otimes \mathfrak{M}_{\eta}$. Since Δ is an algebra map we conclude that

$$\Delta \mathfrak{M}_{\mu \eta} \subseteq \mathfrak{M}_{\mu} \otimes A_p + A_p \otimes \mathfrak{M}_{\eta} \quad \text{or} \quad \mathfrak{M}_{\mu \eta} \subseteq \mathfrak{M}_{\mu} \wedge \mathfrak{M}_{\eta}.$$

Thus

$$(C_{\mu} C_{\eta})^{\perp} = \mathfrak{M}_{\mu} \wedge \mathfrak{M}_{\eta} \supseteq \mathfrak{M}_{\mu \eta} = C_{\mu \eta}^{\perp}$$

which proves $C_{\mu} C_{\eta} \subseteq C_{\mu \eta}$.

(c) is clear.

(d) Let Δ be the antipode of A_p . Then $\Delta(a^n - \eta 1) = a^{-n} - \eta 1 =$

$-\eta a^{-n}(a^n - \eta^{-1}1)$. Thus $\Delta(\mathfrak{M}_\eta) \subseteq \mathfrak{M}_{\eta-1}$. This means $s(C_\eta) = \Delta(\mathfrak{M}_\eta^\perp) \subseteq \mathfrak{M}_{\eta-1}^\perp = C_{\eta-1}$. By (b) C_ρ is easily seen to generate H_p as an algebra, so that the remainder of (d) follows. Q.E.D.

One can easily show that $\text{Rad } A_p = (bz)$ if p is prime to $\text{char } k$. If $k = \mathbf{Q}$ or $\mathbf{R}$ and $n = 2$, then the dual algebra of the simple subcoalgebra of H_2 of dimension > 1 is a division ring.

If $\text{char } k = p$ then $\text{Rad } A_p = (z)$ and H_p is pointed as is A_p . The degrees of nilpotency of $\text{Rad } A_p$ and $\text{Rad } H_p$ are different in this case.

3. The invariance of simple subcoalgebras under $t = s^2$. We will show that for given $n \geq 5$ there exist nonpointed finite-dimensional Hopf algebras with antipode s of order $2n$ such that if C is a simple subcoalgebra of dimension > 1 , then $t^m(C) = C$ implies $n|m$. This is in sharp contrast to [2, 3.3]. Again the Hopf algebras we construct will be generated as algebras by the coradical. The construction is more involved than that of §2, but the details will follow a similar pattern. The basic example will be the dual Hopf algebra of 3.1.

Let $\mathcal{H} = k\{A, X, Y\}$ be the free k -algebra on noncommuting indeterminates A, X , and Y . Then for $q \geq 1$ fixed, $\mathcal{H}$ has a unique bialgebra structure $(\mathcal{H}, \Delta, \epsilon)$ where $\Delta: \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ and $\epsilon: \mathcal{H} \rightarrow k$ are determined by

$$\begin{aligned} \Delta A &= A \otimes A, & \epsilon(A) &= 1, \\ \Delta X &= X \otimes A + 1 \otimes X, & \text{and} & \\ \Delta Y &= Y \otimes A^q + 1 \otimes Y, & \epsilon(X) &= 0 = \epsilon(Y). \end{aligned}$$

3.1. Let $n > 1$ be fixed and assume that k contains a primitive n th root of unity ω . Suppose $q > 2$ and $(q, n) = 1$. Then the ideal γ of $\mathcal{H}$, as above generated by $(p = q - 1)$, $A^{pn} - 1$, $X^n - (A^n - 1)$, $Y^n - (A^n - 1)$, $XA - \omega AX$, $YA - \omega^{-q}AY$, and $XY - \omega^q YX$, is a bi-ideal, and $A_q = \mathcal{H}/\gamma$ is a Hopf algebra with basis $\{a^i x^j y^l : 0 \leq i < pn, 0 \leq j, l < n\}$. Thus $\dim A_q = pn^3$.

To see that γ is a bi-ideal is straightforward except for possibly showing that $\Delta Y^n - (Y^n \otimes A^n + 1 \otimes Y^n) \in \mathcal{H} \otimes \gamma + \gamma \otimes \mathcal{H}$. Now $YA^q \equiv \omega^{-q^2} A^q Y$ and ω^{-q^2} is a primitive n th root of unity since $(q, n) = 1$. Thus by 1.2(b),

$$\Delta Y^n - (Y^n \otimes A^{qn} + 1 \otimes Y^n) \in \mathcal{H} \otimes \gamma + \gamma \otimes \mathcal{H}.$$

But $A^{qn} = A^n A^{n(q-1)} \equiv A^n$ by assumption. As before the dimension statement is straightforward.

Set $g(X) = X^{p-1} + \dots + 1$ and $b = g(a^n)$ as in §2. Then we make the following conclusions about A_q :

3.2. Lemma. Let $\mathfrak{N} = (b)$ and $B = A_q/\mathfrak{N}$ where $p = q - 1 > 1$. Then
(a) $\dim B = (p - 1)n^3$.

Suppose p is prime to $\text{char } k$. Then

(b) B is semisimple.

Assume further that k is algebraically closed and $\rho \in k$ is a primitive p th root of unity. If $\mathfrak{N}_{\rho l} = (a^n - \rho^l 1)$ and $B_{\rho l} = A_q \mathfrak{N}_{\rho l}$, then

$$(c) \quad B \cong B_{\rho} \times \cdots \times B_{\rho^{p-1}} \quad \text{and} \quad B_{\rho l} \cong \underbrace{M(n, k) \times \cdots \times M(n, k)}_{n\text{-times}}$$

for $1 \leq l < p$.

Proof. (a) It is clear by the proof of 2.2(a) that $\dim \mathfrak{N} = n^3$. Thus $\dim B = (p-1)n^3$.

(b) will follow from (c) as does 2.2(b) follow from 2.2(c).

(c) $a^n - \eta 1$ is in the center of A_q . So again $\mathfrak{N} = \bigcap_{l=1}^{p-1} \mathfrak{N}_{\rho l}$ and $B \cong B_{\rho} \times \cdots \times B_{\rho^{p-1}}$. Since $\dim B_{\rho l} \leq n^3$, we conclude $\dim B_{\rho l} = n^3$ for $1 \leq l < p$. By 1.3(b), $B_{\rho l} \cong M(n, k) \times \cdots \times M(n, k)$ (n -times) since $(\text{mod } \mathfrak{N}_{\rho l})$ we have $a^n \equiv \rho^l 1$, $x^n \equiv (\rho^l - 1)1 \equiv y^n$, and $\rho^l - 1 \neq 0$ for $1 \leq l < p$. Q.E.D.

Let $K_q = A_q^*$ where A_q is as above.

3.3. Proposition. Let k be an algebraically closed field containing a primitive n th root of unity, where $n > q > 2$ and $(q, n) = 1$. Let K_q be as above. Then:

(a) The antipode s of K_q has order $2n$.

(b) If $p = q - 1$ is prime to $\text{char } k$, then K_q is not pointed.

(c) Suppose $(p, n) = 1$ also, and that $C \subseteq K_q$ is a simple subcoalgebra, with $\dim C > 1$. If $t = s^2$ then $t^m(C) = C$ implies $n \mid m$.

Proof. (a) Clear by 1.1.

(b) follows by 3.2(c) and duality.

(c) Consider $\eta \in \text{Alg}(A_q, k) = G(K_q)$ determined by $\eta(a) = \omega$ and $\eta(x) = 0 = \eta(y)$. Then it is easy to see that $G(K_q) = \langle \eta \rangle$ and $\mathcal{I} = \bigcap_{l=0}^{n-1} \ker \eta^l$ is generated as an ideal by x and y . If $\mathfrak{N} = (b)$ as in 3.2, then $x^n b = 0 = y^n b$. Thus $\mathfrak{N}\mathcal{I}$ is nilpotent. But $\mathfrak{N} + \mathcal{I} = A_q$ and b is in the center of A_q , so $\mathfrak{N}\mathcal{I} = \mathfrak{N} \cap \mathcal{I}$. Therefore $\mathfrak{N}\mathcal{I} = \text{Rad } A_q$ by 3.2(b).

Let $C \subseteq K_q$ be simple, and $\dim C > 1$. Then $C^\perp = \mathfrak{M}$, where $\mathfrak{M}$ is generated by $a^n - \rho^l 1$ and $y - \xi a^q x^{n-q}$, where ρ is a primitive p th root of unity, and $\xi \in k$ satisfies $(\xi a^q x^{n-q})^n \equiv \rho^l - 1 \pmod{\mathfrak{N}_{\rho l}}$ by 3.2(c) and 1.3(b). Let Δ be the antipode of A_q . We now determine $\Delta^2(\mathfrak{M})$. Clearly $\Delta^2(a^n - \rho^l 1) = a^n - \rho^l 1$. Now $\Delta(x) = -xa^{-1}$ and $\Delta(y) = -ya^{-q}$, so $\Delta^2(x) = axa^{-1} = \omega^{-1}x$ and $\Delta^2(y) = a^q ya^{-q} = \omega^{q^2}y$. Therefore

$$\Delta^2(y - \xi a^q x^{n-q}) = \omega^{q^2}y - \xi a^q (\omega^{-1}x)^{n-q} = \omega^{q^2}(y - \xi \omega^{q-q^2} a^q x^{n-q}).$$

If $\lambda = \omega^{q-q^2}$, then λ is a primitive n th root of unity, since $q - q^2 = -qp$ and n are relatively prime. Let $\mathfrak{M}_\lambda$ denote the maximal ideal generated by

$a^n - \rho^l 1$ and $y = \xi \lambda^k a^q x^{n-q}$. What we have shown ($r = \Delta^2$) is that $\tau(\mathfrak{M}_{\lambda^k}) = \mathfrak{M}_{\lambda^{k+1}}$. Clearly $\mathfrak{M}_{\lambda^k} = \mathfrak{M}_{\lambda^{k'}}$ implies $\lambda^k = \lambda^{k'}$. Since $\tau^m(\mathfrak{M}_{\lambda^k}) = \mathfrak{M}_{\lambda^k}$ if and only if $\iota^m(\mathfrak{M}_{\lambda^k}^\perp) = \mathfrak{M}_{\lambda^k}^\perp$, we have (c). Q.E.D.

The simplest version of 3.3 occurs when $p = 2$, $q = 3$, $n = 5$ and $k = \mathbf{Q}$, or $\text{char } k > 5$.

Remark. It is possible to handle the case of $\text{char } k = p = n$ in §1 using the basic Hopf algebra A of dimension p^2 described as follows: A is generated by a, z as an algebra subject to $a^p = 1$, $z^p - z = 0$, $[a, z] = a^2 - a$; the coalgebra structure is determined by $\Delta a = a \otimes a$ and $\Delta z = z \otimes a + 1 \otimes z$. The details are much more complicated and are best handled using the ideas of [6], but the main ideas parallel those of this paper. In particular it is possible to have finite-dimensional Hopf algebras over a field of characteristic p with $r - 1$ simple subcoalgebras of dimension p^2 , $(r, p) = 1$.

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