OPERATORS QUASISIMILAR TO A NORMAL OPERATOR

CONSTANTIN APOSTOL¹

ABSTRACT. A necessary and sufficient condition for an operator to be quasisimilar to a normal operator is given.

Let X be a complex Banach space and let T be a bounded linear operator acting in X . The aim of this note is to provide a necessary and sufficient condition for T to be quasisimilar to a normal operator acting in a Hilbert space.

This problem was called to the attention of the author by R. G. Douglas.

Before stating our result, recall some basic definitions. Let Y be a complex Banach space and let $A: Y \rightarrow X$ be a bounded linear operator. We shall call A a quasiaffinity if it is injective and has dense range. Let S be a bounded linear operator acting in Y . We shall say that T is quasisimilar with S if there exist two quasiaffinities $A: Y \rightarrow X$, $B: X \rightarrow Y$, such that $TA = AS$, $SB = BT$. For more details see [1]. A finite or countable system $\{X_n\}_{1 \leq n < m}$ of subspaces on X will be called basic if, for any n , the subspaces X_n , $\text{cl } m\{X_k\}_{k \neq n}$ are complementary and $\bigcap_{1 \leq n < m} (\text{cl } m\{X_k\}_{k \geq n}) = \{0\}$ if $m = \infty$.

We shall prove the following

Theorem. *The operator $T \in \mathcal{L}(X)$ is quasisimilar to a normal operator if and only if there exists a basic system $\{X_n\}_{1 \leq n < m}$ of invariant subspaces of T such that $T|_{X_n}$ is similar to some normal operator.*

Proof. *Sufficiency.* Let $\{X_n\}_{1 \leq n < m}$ be a basic system of invariant subspaces of T such that $T|_{X_n}$ is similar to some normal operator S_n acting in the Hilbert space H_n . Since we have

$$\|S_n\| = \|S_n|_{s p}\| = \|(T|_{X_n})|_{s p}\| \leq \|(T|_{X_n})\| \leq \|T\|,$$

we may define the normal operator $S = \bigoplus_{1 \leq n < m} S_n$ acting in $H = \bigoplus_{1 \leq n < m} H_n$. Let $C_n: H_n \rightarrow X_n$ be an invertible operator such that $T|_{X_n} = C_n S_n C_n^{-1}$, and let P_n be the canonical projection of X onto X_n determined by the decomposition $X = X_n + \text{cl } m\{X_k\}_{k \neq n}$. Choosing positive numbers $\alpha_n, \beta_n \neq 0$,

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such that

$$\sum_{1 \leq n < m} \alpha_n \|C_n\| < \infty, \quad \sum_{1 \leq n < m} \beta_n \|C_n^{-1}\| \|P_n\| < \infty,$$

we may define the quasiaffinities $A: H \rightarrow X$, $B: X \rightarrow H$, by the equation

$$A \left(\bigoplus_{1 \leq n < m} h_n \right) = \sum_{1 \leq n < m} \alpha_n C_n h_n, \quad h_n \in H_n,$$

$$Bx = \bigoplus_{1 \leq n < m} \beta_n C_n^{-1} P_n x, \quad x \in X.$$

But it is easy to see that we have $TA = AS$, $SB = BT$; thus T is quasisimilar with T .

Necessity. Suppose that T is quasisimilar to a normal operator S acting in a Hilbert space H . Let $A: H \rightarrow X$, $B: X \rightarrow H$ be two quasiaffinities such that $TA = AS$, $SB = BT$. Since obviously S commutes with BA , applying Fuglede's theorem we infer that S commutes with the spectral measure E , of the modulus of BA . We consider the two possible cases:

- (i) BA is not invertible,
- (ii) BA is invertible.

(i) If BA is not invertible, then 0 is an accumulation point in the spectrum of the modulus of BA ; thus we can find a strictly decreasing sequence $\{a_n\}_{n=1}^{\infty}$, such that $a_1 = \|BA\|$, $\lim_{n \rightarrow \infty} a_n = 0$, $E([a_n, a_{n+1})) \neq 0$. Let us put

$$H_n = E([a_n, a_{n+1}))H, \quad S_n = S|_{H_n}.$$

Since the restriction of BA to H_n is bounded from below by a_n , the restriction of A to H_n will be bounded from below; thus $X_n = AH_n$ is a closed subspace in X . The relations

$$TX_n = TAH_n = ASH_n \subset AH_n = X_n$$

show that X_n is an invariant subspace of T . If we define $C_n: H_n \rightarrow X_n$ to be the restriction of A to H_n , then it is easy to see that we have $T|_{X_n} = C_n S_n C_n^{-1}$, thus $T|_{X_n}$ is similar to a normal operator. Let us put $\sigma_n = [0, a_1) \setminus [a_n, a_{n+1})$. Since obviously we have $\text{cl } m\{X_k\}_{k \neq n} = \text{cl } m\{AE(\sigma_n)H\}$, using the relations

$$\|A(h_n + h'_n)\|^2 \geq \|B\|^{-2} \|BA(h_n + h'_n)\|^2 = \|B\|^{-2} (\|BAh_n\|^2 + \|BAh'_n\|^2)$$

$$\geq \|B\|^{-2} a_n^2 \|h_n\|^2 \geq \|A\|^{-2} \|B\|^{-2} a_n^2 \|Ah_n\|$$

$$(\forall) h_n \in H_n, h'_n \in E(\sigma_n)H,$$

we obtain that the angle between X_n and $\text{cl } m\{X_k\}_{k \neq n}$ is positive and we have

$$X_n + \text{cl } m\{X_k\}_{k \neq n} = \text{cl } m\{AH\} = X, \quad \bigcap_{n=1}^{\infty} (\text{cl } m\{X_k\}_{k \geq n}) = \{0\}.$$

This shows that the system $\{X_n\}_{n=1}^{\infty}$ is basic.

(ii) If BA is invertible, A results to be invertible; thus T is similar with S and we can take the basic system $\{X_k\}_{1 \leq n < m}$, with $m = 2$, $X_1 = X$.

Remark. An examination of the proof of our Theorem shows that we can replace the word "normal" in the statement by either "selfadjoint" or "unitary".

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DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK AT STONY BROOK, STONY BROOK, NEW YORK 11794

Current address: Institute of Mathematics, Romanian Academy of Sciences, Bucharest, Romania