ON DECOMPOSABILITY OF COMPACT PERTURBATIONS OF OPERATORS

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ABSTRACT. Let A be a Hilbert-space operator satisfying the growth condition $\|(z - A)^{-1}\| \leq \exp\{K[\text{dist}(z, J)]^{-s}\}$, $z \notin J$, where J is a C^2 Jordan curve, and $K > 0$, $s \in (0, 1)$ are two constants. Let $T = A + B$ for some $B \in C_p$, $1 \leq p < \infty$. It is shown that T is strongly decomposable if and only if $\sigma(T)$ does not fill the "interior" of J .

1. H. Radjavi and the author [13] showed that if a Hilbert-space operator T is the sum of a unitary operator A and an operator B of a Schatten class C_p ($1 \leq p < \infty$) such that $\sigma(T)$ does not fill the unit disc, then T is strongly decomposable. In the present paper we show that the above statement remains true if

(a) the unit disc is replaced by a domain whose boundary is a C^2 Jordan curve J ,

(b) the unitary operator is replaced by an operator A whose resolvent satisfies the growth condition

$$(1) \quad \|(z - A)^{-1}\| \leq \exp\{K[\text{dist}(z, J)]^{-s}\}, \quad z \notin J,$$

where $s \in (0, 1)$ and $K > 0$ are two constants. (By S^{-1} we mean it exists and is bounded.)

(Recall that (i) an invariant subspace Y of T is called a maximal spectral subspace of T if $M \subseteq Y$ for all invariant subspaces M of T such that $\sigma(T|M) \subseteq \sigma(T|Y)$, (ii) T is said to be decomposable if for every finite open covering $G_1, G_2, \dots, G_n$ of $\sigma(T)$ there is a family $Y_1, Y_2, \dots, Y_n$ of maximal spectral subspaces of T such that $\sigma(T|Y_i) \subseteq \overline{G_i}$, $i = 1, 2, \dots, n$, and $H = Y_1 + Y_2 + \dots + Y_n$ where H is the underlying Hilbert space, (iii) T is said to be strongly decomposable if the restriction of T to an arbitrary maximal spectral subspace of T is again decomposable.)

Let us begin with a detailed introduction. Many authors have proved the existence of a nontrivial invariant subspace for the perturbation of a Hermitian operator A by an operator B of a class C_p ($1 \leq p < \infty$); a complete list of these authors is given in [6, p. 2122]. These results were further generalized by K. Kitano [8] to the case where A is a normal operator

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with spectrum on a C^2 Jordan curve (see also [14, Chapter 6] in this direction) and by C. Apostol [1, Theorem 3.5 and its proof] to the case where the resolvent of A satisfies the growth condition

$$(2) \quad \|(z - A)^{-1}\| \leq K[\text{dist}(z, J)]^{-n}, \quad z \notin J,$$

where J is a C^2 Jordan curve and K, n are positive constants. I. Colojoară and C. Foiaş [3, Theorem 5.5.2, p. 166] showed that if $T^*T - I \in C_p$ ($1 \leq p < \infty$) and $\sigma(T)$ lies on the unit circle, then T is strongly decomposable (actually they showed that T is an $\mathcal{U}$ -unitary operator which is much stronger); the author [12, Corollary 3] proved that if T is the sum of an operator A satisfying (2) and an operator $B \in C_p$ ($1 \leq p < \infty$), and if $\sigma(T) \subseteq J$, then T is strongly decomposable. A. Jafarian [7] showed that if $0 \leq I - T^*T \in C_1$ (the trace class) and $\sigma(T)$ does not fill the unit disc, then T is still strongly decomposable (see also B. Sz. Nagy and C. Foiaş, [16, Theorem VIII. 3.1, p. 337]). More generally, H. Radjavi and the author [13] proved that if $\sigma(T)$ does not fill the interior of the unit disc then the condition $T^*T - I \in C_p$ ($1 \leq p < \infty$) is sufficient for strong decomposability of T . Actually we proved more; if $\sigma(T)$ does not fill the "interior" of a C^2 Jordan curve J and if, for any hyperinvariant subspace M of T , $T|_M$ (respectively $(T^*|_{M^\perp})^*$) is the sum of a normal operator having spectrum on J and an operator of class C_p ($1 \leq p < \infty$), then T is strongly decomposable. As a generalization of these results we prove

Theorem 1. *Let A satisfy the growth condition (1). Let $B \in C_p$ for some $p \in [1, \infty)$ and let $T = A + B$. Then T is strongly decomposable if and only if $\sigma(T)$ does not fill the interior of J .*

The proof is given in the next section. As immediate corollaries we have:

Corollary 1. *If T is the sum of a normal operator with spectrum on a C^2 Jordan curve J and an operator of class C_p ($1 \leq p < \infty$), then T is strongly decomposable if and only if $\sigma(T)$ does not fill the interior of J .*

Corollary 2. *Let T be as in Theorem 1. Assume $\sigma(T)$ has more than one point. Then T has a nontrivial hyperinvariant subspace.*

2. We prove Theorem 1 in a series of propositions and lemmas. All operators are bounded and defined on a Hilbert space.

Notations and definitions. Let $B(H)$ be the space of all bounded linear operators defined on a Hilbert space H .

(a) An operator $T \in B(H)$ is said to have the single valued extension property if there exists no nonzero H -valued analytic function f such that $(z - T)f(z) \equiv 0$.

(b) $\bar{G}$ = the closure of a set $G \subseteq \mathbb{C}$.

(c) C_p is the ideal of compact operators T such that $\sum (\mu_n)^p < \infty$, where $\mu_1, \mu_2, \dots$ are the eigenvalues of $(T^*T)^{1/2}$ arranged in decreasing order and repeated according to multiplicity for $0 < p < \infty$; C_∞ is the ideal of all compact operators.

The following proposition is known.

Proposition 1. *Let $T \in B(H)$ be the sum of an invertible operator A and a compact operator B . Assume $\sigma(T)$ fills a neighbourhood of the origin. Then T is not decomposable.*

Lemma 1. *Let ϕ be a conformal mapping from the unit disc onto the "interior" of a C^2 Jordan curve J . Then there exist two positive constants K_1 and K_2 such that*

$$K_1(1 - |z|) \leq \text{dist}(\phi(z), J) \leq K_2(1 - |z|) \quad \text{for } |z| < 1.$$

Proof. The first inequality follows from the Koebe-Bieberbach one quarter theorem [18, Theorem III. 40, p. 85] (see also [15, proof of Theorem 8, p. 143] for more details); the second inequality follows from [18, Chapter IX, §3, Lemma 1, p. 359] together with the fact that $\phi'(z)$ exists and is uniformly bounded for $|z| \leq 1$ [18, proof of Theorem IX.7, p. 361].

The following proposition is central to our investigations. First we need some definitions.

Let $B \in C_p$ for some positive integer p and let x, y be two arbitrary unit vectors in the underlying Hilbert space H . Let

$$D_0(B) = \prod_i (1 - \lambda_i) \exp \left(\sum_{j=1}^{p-1} j^{-1} (\lambda_i)^j \right), \quad D_1(B) = D_0(B) ((I - B)^{-1} x | y),$$

where $\lambda_1, \lambda_2, \dots$ are the eigenvalues of B , numbered according to decreasing values of their moduli and repeated according to their multiplicities. Define $\|B\|_p = [\sum (\lambda_i)^p]^{1/p}$ if B is positive, and $\|B\|_p = \|(B^*B)^{1/2}\|_p$ in general. It is shown in [5, Lemma XI. 9.22 and Corollary XI. 9.25, p. 1106 and 1112] that if $1 \notin \sigma(B)$, then $\ln |D_i(B)| \leq \Gamma (\|B\|_p)^p$, $i = 0, 1$, where Γ is a positive number depending only on p . (See also [10, Lemma 1].) Moreover, if S is an arbitrary operator on H , then $\|SB\|_p \leq \|S\| \|B\|_p$ and $\|BS\|_p \leq \|S\| \|B\|_p$.

Proposition 2. *Let T be as in Theorem 1 and let $\sigma(T)$ not fill the "interior" of J . Let (the open set) G be either the "interior" or the "exterior" of J . Let J_1 be a nontrivial closed subarc of J . Let M be an invariant subspace of T such that $[G \cap \sigma(T|M)]^- \cap J_1 = \emptyset$. Then there exist a constant $C > 0$ and an open neighbourhood Δ of J_1 such that*

$$(3) \quad \|(z - T|M)^{-1}\| \leq \exp\{\exp[C \operatorname{dist}(z, J)^{-s}]\}$$

for $z \in \Delta \cap G$.

Proof. Assume without loss of generality that p is a positive integer. Let Δ be an open neighbourhood of J_1 such that $\bar{\Delta} \cap G$ contains no points of $\sigma(T|M)$ and the boundary of $\Delta \cap G$ is a C^2 Jordan curve (containing J_1). Let ϕ be a conformal mapping from the unit disc onto $\Delta \cap G$. Let x, y be two unit vectors in M , $f(z) = ((z - T|M)^{-1}x|y)$ for $z \in \Delta \cap G$, and $g(t) = f(\phi(t))$ for $t \in \mathbb{C}$ and $|t| < 1$. Let

$$T(r) = \frac{1}{2\pi} \int_0^{2\pi} \ln^+ |g(re^{i\theta})| d\theta, \quad 0 \leq r < 1,$$

be the Nevanlinna's characteristic function of g [11, p. 12], where $\ln^+ \alpha = \max\{0, \ln \alpha\}$ for $\alpha > 0$. For $z \notin \sigma(T)$ and $z \in G \cap \Delta$ we have

$$f(z) = ((z - T)^{-1}x|y) = ([I - B(z - A)^{-1}]^{-1}x|(\bar{z} - A^*)^{-1}y) = \delta_1(z)/\delta_0(z),$$

where

$$\delta_0(z) = D_0(B(z - A)^{-1})[\phi^{-1}(z)]^{-m}, \quad \delta_1(z) = \delta_0(z)f(z),$$

and m is the order of the zero of $D_0(B(z - A)^{-1})$ at $z = \phi(0)$. (In view of [14, Lemma 6.7] δ_0 , and consequently δ_1 , are analytic on $\mathbb{C} \setminus J$.) It follows from the paragraph preceding the proposition that for $z \notin \sigma(T)$,

$$\ln^+ |\delta_i(z)| \leq \exp\{\Gamma_1[\operatorname{dist}(z, J)]^{-s}\} \leq \exp\{\Gamma_2|1 - \phi^{-1}(z)|^{-s}\}, \quad i = 0, 1,$$

where Γ_1, Γ_2 are positive numbers independent of z (see also Lemma 1). Choose $r \in (0, 1)$ such that $\phi(re^{i\theta}) \notin \sigma(T)$ for all $\theta \in [0, 2\pi]$. This is possible because $\sigma(T) \setminus J$ is countable (see [13, Lemma 1]). Applying Jensen's formula [22, p. 3] to $\delta_0(\phi(t))$ we obtain

$$\begin{aligned} \ln |\delta_0(\phi(0))| &= \frac{1}{2\pi} \int_0^{2\pi} \ln^+ |\delta_0(\phi(re^{i\theta}))| d\theta \\ &\quad - \frac{1}{2\pi} \int_0^{2\pi} \ln^+ \left| \frac{1}{\delta_0(\phi(re^{i\theta}))} \right| d\theta - \sum_k \ln \frac{r}{|a_k|}, \end{aligned}$$

where $a_1, a_2, \dots$ are the roots of $\delta_0(\phi(t))$ in the disc $|t| < r$. Therefore

$$\begin{aligned} T(r) &\leq \frac{1}{2\pi} \int_0^{2\pi} \ln^+ |\delta_1(\phi(re^{i\theta}))| d\theta \\ &\quad + \frac{1}{2\pi} \int_0^{2\pi} \ln^+ |\delta_0(\phi(re^{i\theta}))| d\theta - \ln |\delta_0(\phi(0))| \end{aligned}$$

or $T(r) \leq \exp\{\Gamma_3(1-r)^{-s}\}$ where $\Gamma_3 > 0$ is independent of r . Since the set $\{r \in [0, 1]: \phi(re^{i\theta}) \notin \sigma(T) \text{ for all } \theta \in [0, 2\pi]\}$ is dense in $[0, 1]$, we have

$T(r) \leq \exp\{\Gamma_3(1-r)^{-s}\}$ for all $r \in [0, 1]$. Thus by [11, p. 140],

$$\ln|g(re^{i\theta})| \leq \exp\{\Gamma_4(1-r)^{-s}\} \quad \text{for } 0 \leq r < 1,$$

where Γ_4 is independent of r and θ . Another application of Lemma 1 yields

$$\ln^+ |(z)| \leq \exp\{C[\text{dist}(z, J)]^{-s}\} \quad \text{for } z \in \Delta \cap G,$$

where C is a constant independent of z . (Note that $f(z)$ remains bounded off J .) Since x, y were arbitrary, the proof of the proposition is complete.

Some of the techniques of the proof of Proposition 2 are inspired from the proof of Theorem 1 of [10].

Proof of Theorem 1. The necessity of the condition follows from Proposition 1. The proof of the sufficiency is the same as the proof of [13, Theorem 1] with a slight difference. This difference is in the proof of [13, Lemma 5] which is a step toward the proof of [13, Theorem 1]. The growth condition (3) of Proposition 2 is not exactly the same as the one needed in the proof of [13, Lemma 5]. Fortunately, a similar proof can be constructed by using some results of J. I. Ljubic and V. I. Macaev (see [9, Lemma 2.2.1, p. 119]).

Although we have solved Problem 1 of [13], the following conjecture of that paper remains unsettled.

Conjecture. If T is the sum of a normal operator having spectrum on a C^2 Jordan curve and an operator of some class C_p ($1 \leq p < \infty$), then the same is true for the restriction of T to any hyperinvariant subspace. (The conjecture is true for $p = \infty$ [13, Theorem 2].)

General remark. In view of [6, Theorem XVI.5.18, p. 2162], a spectral operator of finite type having spectrum on a C^2 Jordan curve J satisfies the growth condition (1).

REFERENCES

1. C. Apostol, *On the growth of resolvent, perturbation and invariant subspaces*, Rev. Roumaine Math. Pures Appl. 16 (1971), 161–172. MR 45 #933.
2. J. W. Calkin, *Two-sided ideals and congruences in the ring of bounded operators in Hilbert space*, Ann. of Math. (2) 42 (1941), 839–873. MR 3, 208.
3. I. Colojoară and C. Foiaş, *The theory of generalized spectral operators*, Gordon and Breach, New York, 1968.
4. R. G. Douglas, *Banach algebra techniques in operator theory*, Academic Press, New York, 1972.
5. N. Dunford and J. T. Schwartz, *Linear operators. Part II: Spectral theory. Selfadjoint operators in Hilbert space*, Interscience, New York, 1963. MR 32 #6181.
6. ———, *Linear operators. III*, Interscience, New York, 1971.
7. A. A. Jafarian, *Weak contractions of Sz.-Nagy and Foiaş are decomposable*, Rev. Roumaine Math. Pures Appl. (to appear).
8. K. Kitano, *Invariant subspaces of some non-selfadjoint operators*, Tôhoku Math. J. (2) 20 (1968), 313–322. MR 39 #2001.
9. Ju. I. Ljubič and V. I. Macaev, *Operators with separable spectrum*, Mat. Sb. 56 (98) (1962), 433–468; English transl., Amer. Math. Soc. Transl. (2) 47 (1965), 89–129. MR 25 #2450.

10. V. I. Macaev, *A method for the estimation of the resolvents of non-selfadjoint operators*, Dokl. Akad. Nauk SSSR 154 (1964), 1034–1037 = Soviet Math. Dokl. 5 (1964), 236–240. MR 28 #1495.
11. R. Nevanlinna, *Le théorème de Picard-Borel et la théorie des fonctions méromorphes*, Gauthier-Villars, Paris, 1929.
12. M. Radjabalipour, *Growth conditions and decomposable operators*, Canad. J. Math. 26 (1974), 1372–1379.
13. M. Radjabalipour and H. Radjavi, *On decomposability of compact perturbations of normal operators*, Canad. J. Math. (to appear).
14. H. Radjavi and P. Rosenthal, *Invariant subspaces*, Springer-Verlag, Berlin, 1973.
15. J. G. Stampfli, *A local spectral theory for operators. III: Resolvents, spectral sets and similarity*, Trans. Amer. Math. Soc. 168 (1972), 133–151. MR 45 #4182.
16. B. Sz.-Nagy and C. Foiaş, *Analyse harmonique des opérateurs de l'espace de Hilbert*, Masson, Paris; Akad. Kiadó, Budapest, 1967; English rev. transl., North-Holland, Amsterdam, American Elsevier, New York; Akad. Kiadó, Budapest, 1970. MR 37 #778; 43 #947.
17. A. E. Taylor, *Introduction to functional analysis*, Wiley, New York; Chapman & Hall, London, 1958. MR 20 #5411.
18. M. Tsuji, *Potential theory in modern function theory*, Maruzen, Tokyo, 1959. MR 22 #5712.

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