A CLOSED GRAPH THEOREM

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ABSTRACT. A closed graph theorem is proved, implying that a Hausdorff locally convex space E need not be barrelled if every closed linear map from E into F is continuous, where F is a reflexive Fréchet or LF -space or a space of distributions.

A Hausdorff locally convex space (E, τ) is barrelled if and only if for each Banach space (F, ξ) , a closed linear map from (E, τ) into (F, ξ) is necessarily continuous. This result of M. Mahowald [2, Theorem 2.2] is shown here (Corollary) to be false if instead (F, ξ) is a reflexive Fréchet or LF -space.

Theorem. Let (E, τ) be a Hausdorff locally convex space. Suppose that for a family $(S_\alpha)_{\alpha \in \Phi}$ of absolutely convex compact subsets, there is no locally convex topology on E strictly finer than τ which coincides with τ on each S_α . Let (F, ξ) be a Hausdorff locally convex space in which every closed bounded set is compact. If every closed linear map from any Banach space into (F, ξ) is bounded, then a closed linear map f from (E, τ) into (F, ξ) is necessarily continuous.

Proof. Let E_α denote the linear span of S_α , and τ_α the τ -induced topology on E_α . Let η_α denote the Banach space topology on E_α with the sequence $(S_\alpha/n)_{n=1}^\infty$ of sets as a base of neighborhoods of the origin.

As the graph of the restriction map of f to E_α is closed in $(E_\alpha, \tau_\alpha) \times (F, \xi)$, it is necessarily closed in $(E_\alpha, \eta_\alpha) \times (F, \xi)$ since η_α is finer than τ_α . By the hypothesis then, the closure T_α of $f(S_\alpha)$ in (F, ξ) is compact. The graph G_α of the restriction map f_α of f to S_α is closed in the compact Hausdorff space $X_\alpha = S_\alpha \times T_\alpha$, where S_α, T_α are considered under the τ - and ξ -induced topologies, respectively.

Let $P_{S_\alpha}, P_{T_\alpha}$ be the projection maps of X_α onto S_α, T_α respectively, and P_{G_α} the restriction of P_{S_α} to G_α . As P_{G_α} is continuous and one-to-one, it is necessarily a homeomorphism, since G_α is compact. Therefore each map $f_\alpha = P_{T_\alpha} \circ P_{G_\alpha}^{-1}: S_\alpha \rightarrow (F, \xi)$ is continuous. This implies that $f: (E, \tau) \rightarrow (F, \xi)$ is continuous.

In the Theorem, (E, τ) could be the dual of any complete Hausdorff locally convex space under the topology of compact convergence [1, p. 16,

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Example B], and (F, ξ) could be any Montel Fréchet or LF-space or any of the distribution spaces $\mathcal{E}, \mathcal{E}', \mathcal{D}, \mathcal{D}', \mathcal{T}, \mathcal{T}', \mathcal{O}_M, \mathcal{O}'_M, \mathcal{O}_c, \mathcal{O}'_c$ by [3].

Corollary. *Let (E, τ) be the dual of a complete Hausdorff locally convex space under the Mackey topology τ . If (F, ξ) is a reflexive Fréchet or LF-space or any of the distribution spaces above, then any closed linear map f from (E, τ) into (F, ξ) is necessarily continuous.*

Proof. Let τ_1 denote the topology on E of compact convergence and ξ_0 the weak topology on F associated with ξ . The graph of f is necessarily closed in $(E, \tau_1) \times (F, \xi_0)$, and by the Theorem then $f: (E, \tau_1) \rightarrow (F, \xi_0)$ is continuous. This implies that $f: (E, \tau) \rightarrow (F, \xi)$ is continuous.

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