LINEAR INTERPOLATING BASES IN $C[0, 1]$ ARE NOT BESSELIAN

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ABSTRACT. It is shown that linear interpolating bases for $C[0, 1]$ cannot be p -Besselian.

Introduction. A basis $\{f_i\}$ for a Banach space is a Besselian (p -Besselian) basis iff $\sum a_i f_i$ convergent implies $\{a_i\}$ is in $l_2(l_p)$, and $\{f_i\}$ is a Hilbertian (p -Hilbertian) basis iff $\{a_i\}$ in $l_2(l_p)$ implies $\sum a_i f_i$ converges. In [1] Pełczyński raised the question of the existence of bounded Besselian and bounded Hilbertian bases in $C[0, 1]$. It was shown in [2] that $C[0, 1]$ has a bounded Hilbertian basis. However, the question of the existence of a bounded Besselian basis is still open.

The standard way of constructing a basis for $C[0, 1]$ is to select a dense sequence $\{x_i\}$ in $[0, 1]$ and a corresponding sequence of functions $\{f_i\}$ such that $f_n(x_n) = 1$, $f_n(x_j) = 0$ for $j < n$ and f_n is linear between adjacent points of $\{x_i\}_{i=1}^n$. A generalization of this technique uses a doubly subscripted sequence $\{x_{i,j} : j = 1, \dots, m(i); i = 1, 2, \dots\}$ which is dense in $[0, 1]$ and a corresponding sequence of functions $\{f_{i,j}\}$ satisfying the following three conditions: $f_{i,j}(x_{k,l}) = 0$ for $k < i$; the matrix F_n with i, j th entry $f_{n,j}(x_{n,i})$ is nonsingular; and $f_{i,j}$ is linear between adjacent points of $\{x_{k,l}\}_{k=1}^i$. (Whenever we use either of the symbols $x_{i,j}$ or $f_{i,j}$ in a statement and j is not specified, it will be assumed that j may have any of the values $1, \dots, m(i)$.) A basis for $C[0, 1]$ of the type described above will be called a linear interpolating basis. A linear interpolating basis has the property that if

$$(1) \quad f = \sum_{i=1}^{\infty} \sum_{j=1}^{m(i)} a_{i,j} f_{i,j},$$

then the truncated sum

$$p_n = \sum_{i=1}^n \sum_{j=1}^{m(i)} a_{i,j} f_{i,j}$$

interpolates f at the points $x_{i,j}$, $i = 1, \dots, n$.

Many of the bases for $C[0, 1]$ which have appeared in the literature have been linear interpolating bases. This includes the bounded Hilbertian basis constructed in [2] and the wc_0 basis constructed in [4]. In this paper we show that a bounded linear interpolating basis cannot be p -Besselian.

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Main results. We will assume that we have a fixed bounded linear interpolating basis $\{f_{i,j}\}$ which interpolates at the corresponding points $\{x_{i,j}\}$. For any f in $C[0, 1]$ we have a unique representation (1). Using the matrix F_n defined above and the vector notation

$$\vec{a}_n = (a_{n,1}, \dots, a_{n,m(n)}), \quad \vec{f}_n = (f(x_{n,1}), \dots, f(x_{n,m(n)})),$$

and

$$\vec{p}_{n-1} = (p_{n-1}(x_{n,1}), \dots, p_{n-1}(x_{n,m(n)})),$$

where p_{n-1} is the truncated sum defined above, we can write the equations which determine the coefficients $a_{n,j}$ as

$$F_n \vec{a}_n = \vec{f}_n \rightarrow \vec{p}_{n-1}.$$

To prove that the basis $\{f_{i,j}\}$ is not p -Besselian we construct a function f in $C[0, 1]$ as in (1) and such that there is a subsequence $\{a_{n_i,1}\}$ of the $\{a_{i,j}\}$ satisfying $|a_{n_i,1}| > \frac{1}{2} \sqrt[p]{i}$ for $i = 1, 2, \dots$. This will imply that $\{a_{i,j}\}$ is not in l_p . The function f will be defined as the limit of a sequence $\{g_i\}$. We will define the sequence $\{g_i\}$ inductively in such a way that each g_i is piecewise linear and for each $n = 1, 2, \dots$, $g_{n+1}(x_{i,j}) = g_n(x_{i,j})$ for i less than a certain number $q(n)$. Letting $w = q(n+1)$, it will follow from the above conditions on the sequence $\{g_i\}$ that

$$\vec{p}_{w-1} = (g_n(x_{w,1}), \dots, g_n(x_{w,m(w)}))$$

and

$$\vec{f}_w = (g_{n+1}(x_{w,1}), \dots, g_{n+1}(x_{w,m(w)})).$$

Since the coefficients $a_{w,1}, \dots, a_{w,m(w)}$ are given by $\vec{a}_w = F_w^{-1}(\vec{f}_w - \vec{p}_{w-1})$, they will be determined by the value of g_{n+1} at the points $x_{w,1}, \dots, x_{w,m(w)}$.

The matrix F_w^{-1} is the transpose of

$$\frac{1}{\det(F_w)} \begin{pmatrix} \text{co}(b_{1,1}) & \dots & \text{co}(b_{1,m(w)}) \\ \vdots & & \vdots \\ \text{co}(b_{m(w),1}) & \dots & \text{co}(b_{m(w),m(w)}) \end{pmatrix}$$

where $b_{i,j}$ is the (i, j) th entry of F_w and $\text{co}(b_{i,j})$ is its cofactor. It is tempting to try and define g_{n+1} in such a way that

$$\vec{f}_w - \vec{p}_{w-1} = (\sqrt[p]{w+1})^{-1} (b_{1,1}, \dots, b_{m(w),1}).$$

If this condition could be realized we would have $a_{w,1} = 1/\sqrt[p]{w+1}$. This definition of g_{n+1} is, however, incompatible with the convergence of the sequence $\{g_i\}$. What we are able to define is a uniformly convergent sequence $\{g_i\}$ such that there is a sequence $q: \mathbb{N} \rightarrow \mathbb{N}$ with

$$\vec{f}_{q(i)} - \vec{p}_{q(i)-1} = (c_1, \dots, c_{m(q(i))}),$$

where $c_j = 0$ or $\pm b_{j,1}$, and

$$|a_{q(i),1}| = \left| \frac{1}{\det(F_{q(i)})} \sum c_j \operatorname{co}(b_{j,1}) \right| > \frac{1}{2\sqrt[p]{i+1}}.$$

This inequality is sufficient to imply $\{a_{i,j}\} \notin l_p$.

The remainder of the paper consists of the statement and proof of our Theorem. The proof is a rigorous development of the ideas outlined above.

Theorem. *The bounded linear interpolating basis $\{f_{i,j}\}$ is not p -Besselian.*

Proof. Since the basis is bounded we can assume $|f_{i,j}| < 1$ for all i and j . To define the sequence $\{g_i\}$, which we discussed above, requires two auxiliary sequences: a sequence $q: \mathbb{N} \rightarrow \mathbb{N}$ and a sequence $h: \mathbb{N} \rightarrow \mathbb{N}$. Define $q(i) = i$ and $h(i) = 0$ for all i such that $2/\sqrt[p]{i} \geq 1$. For i such that $2/\sqrt[p]{i} < 1$, define $h(i)$ to be the largest integer such that $2/\sqrt[p]{i} < 2^{-h(i)}$. We will use the sequence h to guarantee the uniform convergence of $\{g_i\}$ by defining the g_i in such a way that $\|g_i - g_j\| < 2^{-h(i)+1}$ for $j > i$ and $i = 1, 2, \dots$.

To begin, define $g_i = 0$ for i such that $2/\sqrt[p]{i} \geq 1$. Now assume $2/\sqrt[p]{n} < 1$ and g_i and q_i have been defined for $i \leq n$. We consider two cases.

Case 1 is $h(n+1) = h(n)$. In this case define $q(n+1) = q(n) + 1$.

Case 2 is $h(n+1) > h(n)$. In this case pick $N > q(n)$ such that if x and y are two adjacent points of $\{x_{i,j} : i < N\}$ and $x < y$, then $V_x^y(g_n) < 2^{-h(n+1)}$. ($V_x^y(g)$ denotes the total variation of the function g on $[x, y]$.) Define $q(n+1) = N$.

In either of the above cases, let $w = q(n+1)$ and define g_{n+1} as follows. Let $I(j)$ be the interval of the form $(k/2^{h(n+1)}, (k+1)/2^{h(n+1)}]$, where $-2^{h(n+1)} \leq k \leq 2^{h(n+1)} - 1$ and $g_n(x_{w,j})$ is in $I(j)$. Choose $e_j = \pm 1$; $j = 1, \dots, m(w)$ such that $g_n(x_{w,j}) + e_j/\sqrt[p]{n+1}$ is in $I(j)$. Consider the cofactor expansion of $\det(F_w)$:

$$\det(F_w) = \sum_{j=1}^{m(w)} b_j \operatorname{co}(b_j)$$

where $b_j = f_{w,1}(x_{w,j})$. We can write this expansion as

$$\det(F_w) = \sum_{j=1}^{m(w)} \frac{1+e_j}{2} b_j \operatorname{co}(b_j) + \sum_{j=1}^{m(w)} \frac{1-e_j}{2} b_j \operatorname{co}(b_j).$$

We consider two cases. Either (Case A)

$$\left| \sum_{j=1}^{m(w)} \frac{1+e_j}{2} b_j \operatorname{co}(b_j) \right| \geq |\det(F_w)|/2$$

or (Case B) the above inequality does not hold and

$$\left| \sum_{j=1}^{m(w)} \frac{1-e_j}{2} b_j \operatorname{co}(b_j) \right| \geq |\det(F_w)|/2.$$

In Case A define c_j , $j = 1, \dots, m(w)$, by $c_j = ((e_j + 1)/2\sqrt[p]{n+1})b_j$. In Case B define c_j , $j = 1, \dots, m(w)$, by $c_j = ((e_j - 1)/2\sqrt[p]{n+1})b_j$. Now define $g_{n+1}(x_{w,j}) = g_n(x_{w,j}) + c_j$ for $j = 1, \dots, m(w)$. For $i < w$ define $g_{n+1}(x_{i,j}) = g_n(x_{i,j})$. We complete the definition of g_{n+1} by defining it to be linear between adjacent points of $\{x_{i,j} : i \leq w\}$. To see that the sequence $\{g_n\}$ is Cauchy, note that for $m > n$ $\|g_m - g_n\|_\infty < 2^{-h(n)+1}$. Define $f = \lim g_n$.

Now fix a positive integer n and let $w = q(n+1)$. The function

$$p_{w-1} = \sum_{i=1}^{w-1} \sum_{j=1}^{m(i)} a_{i,j} f_{i,j}$$

interpolates f at the points $\{x_{i,j} : i < w\}$ and is linear between any two of these points which are adjacent. The function g_n also satisfies these conditions. Hence $p_{w-1} = g_n$.

We noted earlier that the coefficients in the expansion of f are given by $\vec{a}_i = F_i^{-1}(\vec{f}_i - \vec{p}_{i-1})$. In particular, letting $b_j = f_{w,1}(x_{w,j})$, we have

$$\begin{aligned} a_{w,1} &= \frac{1}{\det(F_w)} \sum_{j=1}^{m(w)} \text{co}(b_j)(f(x_{w,j}) - p_{w-1}(x_{w,j})) \\ &= \frac{1}{\det(F_w)} \sum_{j=1}^{m(w)} \text{co}(b_j)(g_{n+1}(x_{w,j}) - g_n(x_{w,j})). \end{aligned}$$

The factor $(g_{n+1}(x_{w,j}) - g_n(x_{w,j}))$ is either $((e_j + 1)/2\sqrt[p]{n+1})b_j$ or $((e_j - 1)/2\sqrt[p]{n+1})b_j$, depending on whether Case A or Case B was used in the definition of g_{n+1} . In either case, it follows from the definition of g_{n+1} that $|a_{w,1}| > 1/2\sqrt[p]{n+1}$. This inequality implies the basis $\{f_{i,j}\}$ is not p -Besselian.

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