ON ARNOLD'S FORMULA FOR THE DIMENSION OF A POLYNOMIAL RING

PAUL EAKIN¹

ABSTRACT. If R is a commutative integral domain with quotient field K and $x_1, \dots, x_n$ are indeterminates, then there exist $\theta_1, \dots, \theta_n$ in K such that $\dim R[x_1, \dots, x_n] = n + \dim R[\theta_1, \dots, \theta_n]$.

If R is a commutative ring, the *Krull dimension* of R is the maximum of the lengths of all chains of prime ideals in R . If $R = \mathcal{C}[V]$ is the coordinate ring of an affine variety V over the complex numbers, then increasing chains of primes in R correspond to decreasing chains of irreducible subvarieties. In this "geometric case" the Krull dimension corresponds to our intuitive notion of (complex) topological dimension. Moreover, since $R[X]$ corresponds to $V \times \mathcal{C}$ (the product of V and an affine line), intuition would lead us to suspect

$$(*) \quad \dim R[X] = \dim R + 1.$$

In [7], W. Krull established $(*)$ for any noetherian ring. Seidenberg [9], [10] investigated the validity of $(*)$ for arbitrary commutative rings and observed that it does not hold in general. He observes that one always has

$$\dim R + 1 \leq \dim R[X] \leq 2 \dim R + 1,$$

and he provides examples to show that within these bounds anything can happen.

Jaffard [6] made an extensive study of the dimension theory in polynomial rings. He introduced the notion of *valuative dimension* of a domain R . This is just the maximum of the ranks of the valuation overrings of R . Jaffard showed that when $(*)$ fails, the valuative dimension of R must exceed the dimension of R . In addition, he studied the asymptotic behavior of the function $f(n) = \dim R[X_1, \dots, X_n]$ and showed that if R is a domain of finite valuative dimension, then for all suitably large n one has $f(n+1) = f(n) + 1$.

In [4] Gilmer and Bastida call the sequence $\{f(i)\}_{i=0}^{\infty}$ the *dimension sequence* of the ring R , and they investigate which sequences are dimension sequences of a certain class of rings. In [2] Arnold and Gilmer determine all sequences which are the dimension sequence of a commutative ring.

Both [2] and [4] depend upon a result of Arnold [1, Theorem 5, p. 323] which we refer to as Arnold's formula. We state the result as follows:

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If R is a commutative integral domain with quotient field K and $X_1, \dots, X_n$ are indeterminants over R , then there exist $\theta_1, \dots, \theta_n$ in K such that

$$(**) \quad \dim R[X_1, \dots, X_n] = n + \dim R[\theta_1, \dots, \theta_n].$$

One always has that in $(**)$ the left-hand side is greater than or equal to the right-hand side. Thus the interesting fact is that the maximum possible dimension of the rings of the form $R[\theta_1, \dots, \theta_n]$ can always be realized. The proof of the formula in [1] is, however, incorrect and we know of no correct proof in the literature.^{2,3} Our purpose here is to provide an elementary proof.

In what follows, all rings are assumed to be commutative and to possess an identity. When we write “ $\dim R$ ” we are referring to the Krull dimension of the ring R . By $R[X_1, \dots, X_n]$ we denote the ring of polynomials in the independent variables $\{X_1, \dots, X_n\}$ over the ring R . Finally, whenever we use the symbol “ $<$ ” it is meant to denote strict containment.

Our argument requires a few well-known facts which we list for the convenience of the reader.

(A) Let k be a field and $R = k[x_1, \dots, x_n]$ a finitely generated ring extension of k . If P is a prime of R , then $\dim R = \text{rank } P + \text{trans deg}_k (R/P)$ [8, p.46, (14.6)].

The following is observed by Seidenberg [9] and is a consequence of (A) applied to the ring of polynomials in n variables over a field.

(B) If $\{P_i\}_{i=0}^l$ is a chain of primes in $R[X_1, \dots, X_n]$ all of which lie over the same prime of R , then $l \leq n$.

(C) If V^* is a valuation ring of rank n with quotient field L , and if L is of transcendence degree d over the field K , then $\text{rank } (V^* \cap K) \geq n - d$ [3, p.440, Corollary 2].

A chain $\mathfrak{D} = \{P_i\}_{i=0}^m$ of primes in a polynomial ring $R[X_1, \dots, X_n]$ is called a *special chain* if, for each $P_i \in \mathfrak{D}$, the ideal $(P_i \cap R)[X_1, \dots, X_n]$ is a member of $\mathfrak{D}$.

(D) JAFFARD'S SPECIAL CHAIN THEOREM. If Q is a prime ideal of $R[X_1, \dots, X_n]$ of finite rank $r(Q)$, then $r(Q)$ can be realized as the length of a special chain of primes in $R[X_1, \dots, X_n]$ with terminal element Q . In particular, if R is finite dimensional then $\dim R[X_1, \dots, X_n]$ can be realized as the length of a special chain of primes of $R[X_1, \dots, X_n]$.

This is the statement of Jaffard's theorem in [5]. The authors give an argument there which we feel is much easier than Jaffard's proof [6].

Proof of Arnold's formula. Suppose there were a counterexample, say $R[X_1, \dots, X_n]$. Then we may assume that n is minimal and that for this fixed n , R has minimal dimension. We must have that both n and $\dim R$ are greater than zero: if $\dim R = 0$, R is a field and by (A) we could simply take θ 's = 0. If $n = 0$ we could again take all θ 's = 0.

² I wish to thank Jon Johnson for bringing the error to my attention.

³ Bastida and Gilmer point out the error in [4]. However their discussion of the mistake is itself erroneous. In [4] they claim to remove the doubt about this formula by independently proving Theorem 3 of [1] (which has the incorrect proof). However their argument ultimately rests on Theorem 5 of [1] whose proof (in [1]) is based on Theorem 3 of [1].

Let $s = \dim R[X_1, \dots, X_n]$. By (D) there is a special chain $0 < P_1 < \dots < P_t < \dots < P_s$ in $R[X_1, \dots, X_n]$. Let t be minimal such that $P_t \cap R \neq 0$. Let $P_t \cap R = q$. By (A), $t - 1 \leq n$ and $P_t = q[X_1, \dots, X_n]$ by the minimality of t . Since our chain has maximal length, $\text{rank } q[X_1, \dots, X_n] = t$. Let $S = R \setminus q$ and localize $R[X_1, \dots, X_n]$ at S to get $T = R_q[X_1, \dots, X_n]$. Now for any local ring R_q , it is an easy consequence of (D) that $\dim R_q[X_1, \dots, X_n] = n + \text{rank } q[X_1, \dots, X_n]$ (or see [5]). Thus $\dim T = n + t$.

Let $\bar{T} = R_q[X_1, \dots, X_n]/P_{t-1}R_q[X_1, \dots, X_n]$. Then $\dim \bar{T} = n + 1$ and there exists a chain of primes

$$(***) \quad 0 < q\bar{T} < Q_2 < \dots < Q_{n+t} < \bar{T}.$$

Since $P_{t-1} \cap R = 0$ we may identify R with its image in $\bar{T}$ and assume that K , the quotient field of R , is contained in that of $\bar{T}$. Let V^* be a valuation overring of $\bar{T}$ which is centered on the chain (***) [8, p.37, (11.9)]. Let $V = V^* \cap K$.

Claim. $\text{rank } V \geq t$.

Proof of Claim. We first compute the transcendence degree of $\bar{T}$ over R . To do this we may localize $\bar{T}$ at $s' = R \setminus 0$. By the permutability of residue class ring and quotient ring formation

$$\begin{aligned} \bar{T}_{s'} &\cong (R_q[X_1, \dots, X_n]/P_{t-1}R_q[X_1, \dots, X_n])_{s'} \\ &\cong K[X_1, \dots, X_n]/P_{t-1}K[X_1, \dots, X_n]. \end{aligned}$$

Since $\text{rank } P_{t-1} = \text{rank } P_{t-1}(K[X_1, \dots, X_n]) = t - 1$, we have $\dim \bar{T}_{s'} = [n - (t - 1)]$ (by (A)). Thus by (A) the transcendence degree of $\bar{T}$ over R is $[n - (t - 1)]$. Therefore we have by (C) that $\text{rank}(V) \geq (n + 1) - [n - (t - 1)] = t$. This establishes the Claim.

Since each prime of the chain (***) contains q , it follows that each prime of V contains q . Thus each prime of V meets R_q at qR_q . Let $M_1 < \dots < M_t$ be a chain in V such that $M_j \cap R_q = qR_q$. For each j such that $1 \leq j \leq t - 1$, choose $\theta_j \in M_{j+1} \setminus M_j$.

Consider the canonical homomorphism $V \xrightarrow{\sigma} V/M_1$. Under σ , $R_q[\theta_1, \dots, \theta_{t-1}]$ maps into V/M_1 . Denote the field R_q/qR_q by k and let $\sigma(\theta_j) = \bar{\theta}_j$. Then under σ we have

$$R/q[\bar{\theta}_1, \dots, \bar{\theta}_{t-1}] \subset k[\bar{\theta}_1, \dots, \bar{\theta}_{t-1}] \subset V/M_1.$$

By our choice of the θ_j 's, the primes M_j/M_1 lie over distinct primes of $k[\bar{\theta}_1, \dots, \bar{\theta}_{t-1}]$ for $j = 1, \dots, t$. Thus $\dim k[\bar{\theta}_1, \dots, \bar{\theta}_{t-1}] \geq t - 1$ and by (A) the elements $\bar{\theta}_1, \dots, \bar{\theta}_{t-1}$ are algebraically independent over k .

Let I denote the kernel of the homomorphism

$$R[X_1, \dots, X_{t-1}] \rightarrow R[\theta_1, \dots, \theta_{t-1}]$$

given by $X_j \rightarrow \theta_j$. If we follow τ by σ we get a mapping

$$R[X_1, \dots, X_{t-1}] \xrightarrow{\sigma \circ \tau} (R/q)[\bar{\theta}_1, \dots, \bar{\theta}_{t-1}] \cong (R/q)[X_1, \dots, X_{t-1}]$$

which clearly has kernel $q[X_1, \dots, X_{t-1}]$. Thus $I \subset q[X_1, \dots, X_{t-1}]$. Moreover, since $I \cap R = 0$, the containment is strict. Again using the fact that

$I \cap R = 0$ we localize the exact sequence

$$0 \rightarrow I \rightarrow R[X_1, \dots, X_{t-1}] \xrightarrow{\tau} R[\theta_1, \dots, \theta_{t-1}] \rightarrow 0$$

at the multiplicative system $R \setminus 0$. This yields an exact sequence

$$0 \rightarrow I(K[X_1, \dots, X_{t-1}]) \rightarrow K[X_1, \dots, X_{t-1}] \rightarrow K \rightarrow 0.$$

Thus $I \cdot (K[X_1, \dots, X_{t-1}])$ has rank $t - 1$ and it follows that I has rank $t - 1$. Recalling that $t - 1 \leq n$, we have now shown that $P_t = q[X_1, \dots, X_n]$ contains the prime $I[X_t, \dots, X_n]$ which is the kernel of the homomorphism

$$R[X_1, \dots, X_n] \rightarrow R[\theta_1, \dots, \theta_{t-1}][X_t, \dots, X_n]$$

which takes X_i to θ_i if $1 \leq i \leq t - 1$ and X_j to X_j if $j \geq t$. Since

$$t - 1 = \text{rank } I \leq \text{rank } I[X_t, \dots, X_n] < \text{rank } q[X_1, \dots, X_n] = t,$$

we see that $\text{rank } I[X_t, \dots, X_n] = t - 1$. We can now modify our original chain $0 < P_1 < \dots < P_t < \dots < P_n$ so that $P_{t-1} = I[X_t, \dots, X_n]$. Hence we have the computation

$$\begin{aligned} [s - (t - 1)] &= \dim R[X_1, \dots, X_n]/P_{t-1} = \dim R[X_1, \dots, X_n]/I[X_t, \dots, X_n] \\ &= \dim R[\theta_1, \dots, \theta_{t-1}][X_t, \dots, X_n]. \end{aligned}$$

There are now two possibilities, each of which leads to a contradiction.

Case 1. $t > 1$. In this case, by the minimality of n there exist $\gamma_t, \dots, \gamma_n$ in K such that

$$\begin{aligned} s - (t - 1) &= \dim R[\theta_1, \dots, \theta_{t-1}, X_t, \dots, X_n] \\ &= [n - (t - 1)] + \dim R[\theta_1, \dots, \theta_{t-1}, \gamma_t, \dots, \gamma_n]. \end{aligned}$$

That is,

$$\dim R[\theta_1, \dots, \theta_{t-1}, \gamma_t, \dots, \gamma_n] + n = s = \dim R[X_1, \dots, X_n].$$

This is a contradiction.

Case 2. $t = 1$. In this case $P_1 = q[X_1, \dots, X_n]$ and we have

$$s - 1 = \dim R[X_1, \dots, X_n]/q[X_1, \dots, X_n] = \dim R/q[X_1, \dots, X_n].$$

By the minimality of $\dim R$, there are $\bar{\theta}_1, \dots, \bar{\theta}_n$ in the quotient field of R/q such that

$$\dim(R/q)[X_1, \dots, X_n] = \dim(R/q)[\bar{\theta}_1, \dots, \bar{\theta}_n] + n.$$

Let $\lambda: R \rightarrow R/q$ be the canonical homomorphism. Then λ extends to a mapping of R_q onto the quotient field of R/q . Choose $\theta_1, \dots, \theta_n$ in R_q such that $\lambda(\theta_i) = \bar{\theta}_i$. Then we have an induced homomorphism

$$R[\theta_1, \dots, \theta_n] \xrightarrow{\lambda} (R/q)[\bar{\theta}_1, \dots, \bar{\theta}_n].$$

Since this homomorphism has a nontrivial kernel,

$$\begin{aligned}\dim R[\theta_1, \dots, \theta_n] &\geq \dim(R/q)[\bar{\theta}_1, \dots, \bar{\theta}_n] + 1 \\ &= \dim(R/q)[X_1, \dots, X_n] - n + 1 = s - 1 - n + 1 = s - n.\end{aligned}$$

Thus

$$\dim R[\theta_1, \dots, \theta_n] + n \geq \dim R[X_1, \dots, X_n].$$

Thus to complete our argument we need only show that $\dim R[\theta_1, \dots, \theta_n] + n \leq \dim R[X_1, \dots, X_n]$. But, as mentioned earlier, this is true for any set of θ 's. For consider the exact sequence

$$0 \rightarrow J \rightarrow R[X_1, \dots, X_n] \rightarrow R[\theta_1, \dots, \theta_n] \rightarrow 0$$

given by the homomorphism $X_i \rightarrow \theta_i$. We surely have $\dim R[X_1, \dots, X_n] \geq \dim R[\theta_1, \dots, \theta_n] + \text{rank } J$. We need only show that $\text{rank } J = n$. Since $J \cap R = 0$, we may localize at the multiplicative system $R \setminus 0 = S$ and compute the rank of J_s . After localizing we have

$$0 \rightarrow J_s \rightarrow K[X_1, \dots, X_n] \rightarrow K \rightarrow 0.$$

If we now apply (A) we compute $\text{rank } J_s = n$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KENTUCKY, LEXINGTON, KENTUCKY 40506