INVERSE THEOREMS FOR FUNCTIONS IN L_p AND OTHER SPACES¹

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ABSTRACT. Inverse theorem proved by R. De Vore for $C(-\infty, \infty)$ is proved to be true also for L_p . This is an answer to a problem posed by De Vore in his book, *The approximation of continuous functions by positive linear operators*.

1. Introduction. R. De Vore [1, Chapter 8] proved that $\|\Delta_{h_n}^2 f\|_C = O(h_n^\alpha)$, $h_n \rightarrow 0+$, and $\|\Delta_h^2 f\|_C = O(h^\alpha)$, $h \rightarrow 0+$, are equivalent provided that $1 \leq h_n/h_{n+1} \leq M < \infty$, where $\Delta_h^2 f(x) = f(x-h) - 2f(x) + f(x+h)$.

G. Freud [2], using the Hausdorff-Young inequality, proved an analogous result for L_2 . De Vore mentions [1, p. 277] that the corresponding L_p problem is open. In this paper it will be shown that $\|\Delta_{h_n}^2 f\|_{L_p} = O(h_n^\alpha)$ and $\|\Delta_h^2 f\|_{L_p} = O(h^\alpha)$ are equivalent, and similar equivalence relations hold for some other spaces. In addition, De Vore's inverse theorem for positive approximating convolution operators on $C[-\pi, \pi]$ is generalized to L_p .

2. L_p result for symmetric differences. The basic simple idea of this paper is given already in the proof of the following theorem:

THEOREM 1. For $f \in L_p(-\infty, \infty)$, or $f \in L_p(-\pi, \pi)$, $1 \leq p < \infty$, the relations $\|\Delta_{h_n}^2 f(\cdot)\|_p = O(h_n^\alpha)$, $h_n \rightarrow 0+$, and $\|\Delta_h^2 f(\cdot)\|_p = O(h^\alpha)$, $h \rightarrow 0+$, are equivalent, where $1 \leq h_n/h_{n+1} \leq M < \infty$, and $0 < \alpha < 2$.

PROOF. Define $F(x) = \int f(x+t)g(t)dt$ for $f \in L_p$, $g \in L_q$ ($q^{-1} + p^{-1} = 1$), and $\|g\|_q = 1$. Since $F(x)$ is continuous, $\|F\|_C = \|F\|_\infty$ and therefore, $\|F\|_C \leq \|f\|_p \|g\|_q$ and

$$\|\Delta_{h_n}^2 F\|_C \leq \|\Delta_{h_n}^2 f\|_p \cdot \|g\|_q \leq \|\Delta_{h_n}^2 f\|_p \leq Kh_n^\alpha.$$

De Vore's theorem implies $\|\Delta_h^2 F\|_C \leq K_1 h^\alpha$ where K_1 depends on α , K and $\{h_n\}$ only. Therefore, $|F(-h) - 2F(0) + F(h)| \leq K_1 h^\alpha$. For every h , we can choose g_h with $\|g_h\|_q = 1$ and

$$|F(-h) - 2F(0) + F(h)| = \left| \int \Delta_h^2 f(t)g_h(t)dt \right| = \|\Delta_h^2 f\|_p \leq K_1 h^\alpha.$$

Theorem 1 can be combined with the necessity of the condition $1 \leq h_n/h_{n+1} \leq M$ proved by G. Freud [2] in the following theorem:

THEOREM 1a. If $0 < \alpha < 2$, $1 \leq p < \infty$ and $h_n \downarrow 0$, then a necessary and sufficient condition for $\|\Delta_{h_n}^2 f\|_p = O(h_n^\alpha)$, $n \rightarrow \infty$, to imply $\|\Delta_h^2 f\|_p = O(h^\alpha)$,

Received by the editors October 14, 1974.

AMS (MOS) subject classifications (1970). Primary 41A25.

¹ This research was supported by NRC Grant A4816.

$h \rightarrow 0+$, is that there exists a constant $M < \infty$ such that $h_n/h_{n+1} \leq M$, $n = 1, 2, \dots$.

3. The result for semigroups.

THEOREM 2. For a C_0 (strongly continuous) semigroup $T(t)$ on a Banach space B ,

$$\|T(t_0)f - 2T(t_0 + h)f + T(t_0 + 2h)f\|_B = \|\Delta_h^2 T(t_0)f\|_B = O(h^\alpha),$$

$$h \rightarrow 0+,$$

and

$$\|\Delta_{h_n}^2 T(t_0)f\|_B = O(h_n^\alpha), \quad h_n \rightarrow 0,$$

are equivalent for a fixed $t_0 \geq 0$, $0 < \alpha < 2$ and $1 \leq h_n/h_{n+1} \leq M < \infty$.

PROOF. Define $F(t) = \langle T(t)f, g \rangle$ where $g \in B^*$ (the dual of B) and $\|g\|_{B^*} = 1$. Now, $\|\Delta_{h_n}^2 T(t_0)f\|_B \leq Kh_n^\alpha$ implies $\|\Delta_{h_n}^2 T(t)\|_B \leq K_1 h_n^\alpha$, $t_0 \leq t \leq t_0 + 1$, and therefore, $|\Delta_{h_n}^2 F(t)| \leq K_1 h_n^\alpha$, $t_0 \leq t \leq t_0 + 1$. Using De Vore's result [1, Theorem 8.3, p. 258], we have $|\Delta_h^2 F(t)| \leq K_2 h^\alpha$, $t_0 \leq t \leq t_0 + 2h$ and, therefore, $|\Delta_h^2 F(t_0)| \leq K_2 h^\alpha$. By choosing for any fixed h an element $g_{h,k} \in B^*$ such that $\|g_{h,k}\| = 1$ and $|\langle \Delta_h^2 T(t_0)f, g \rangle - \|\Delta_h^2 T(t_0)f\|| < 1/k$, the proof is easily completed.

Examples of the applicability of the above theorem are the spaces $B = L_p(0, \infty)$ or $B = L_{p,q}(0, \infty)$, the Lorentz spaces, for which $p \neq \infty$ and $q \neq \infty$ (for definition, see Stein and Weiss [3, p. 191]).

4. Inverse theorem for $L_p(-\pi, \pi)$. Define the convolution approximating operator $L_n f$ by

$$(4.1) \quad L_n f(x) = \int_{-\pi}^{\pi} f(x-t) d\mu_n(t)$$

where $\mu_n(t)$ is increasing, $\mu(-\pi) = 1 - \mu(\pi) = 0$ and $d\mu_n(t)$ is a symmetric measure satisfying

$$(4.2) \quad \int_{-\pi}^{\pi} t^2 d\mu_n(t) = \phi_n^2 = O(1), \quad n \rightarrow \infty, \quad C_1 < \phi_{n+1}/\phi_n < C_2$$

and for every $\varepsilon > 0$, there exists A such that

$$(4.3) \quad \int_{A\phi_n}^{\pi} t^2 d\mu_n(t) \leq \frac{\varepsilon}{2} \phi_n^2.$$

In particular, (4.3) holds if

$$(4.4) \quad \int_{-\pi}^{\pi} t^4 d\mu_n(t) = O(\phi_n^4), \quad n \rightarrow \infty.$$

THEOREM 3. Suppose $f \in L_p(-\pi, \pi)$, $f(x) = f(x + 2\pi)$, $L_n f$ and ϕ_n satisfying (4.1), (4.2) and (4.3); then $\|L_n f - f\|_p = O(\phi_n^\alpha)$ and $\|\Delta_h^2 f\|_p = O(h^\alpha)$ are equivalent for $0 < \alpha < 2$.

PROOF. Define $F(x) = \int_{-\pi}^{\pi} f(x-t)g(t)dt$ where $\|g\|_q = 1$. Obviously $(L_n f(\cdot)) * g(x) = L_n F(x)$ and, therefore,

$$\begin{aligned}\|L_n F - F\|_C &= \|L_n f * g - f * g\|_C = \|(L_n f - f) * g\|_C \\ &\leq \|L_n f - f\|_p \|g\|_q = \|L_n f - f\|_p = O(\phi_n^\alpha).\end{aligned}$$

Using De Vore's result, $\|\Delta_h^2 F(t)\|_C = O(h^\alpha)$ and, following earlier arguments, $\|\Delta_h^2 f\|_{L_p} = O(h^\alpha)$. The other direction is an easy calculation.

5. **Remark.** In his paper [2] G. Freud proves a more general result for $L_2(-\pi, \pi)$ than Theorem 1, namely for h_n as in Theorem 1, $\|\Delta_{h_n}^k f\|_{L_2} \leq O(h_n^\alpha)$ if and only if $\|\Delta_h^k\|_{L_2} = O(h^\alpha)$ for $0 < \alpha < k$. I feel that the same result would be true for L_p and C , but the method used here does not yield it.

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