THE ASYMPTOTIC EXPANSION OF THE ZETA-FUNCTION OF A COMPACT SEMISIMPLE LIE GROUP

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ABSTRACT. If G is a connected, simply connected, semisimple Lie group with metric given by the negative of the Killing form and zeta-function $Z(t)$, then

$$Z(t) = \frac{\text{Vol } G}{(4\pi t)^{\dim G/2}} \exp|\delta|^2 t + \text{exponentially small error as } t \downarrow 0.$$

0. Introduction. Let G be a compact, connected, simply connected, semisimple Lie group with Lie algebra $\mathfrak{g}$. As in [2] the metric g on G will be obtained by left translation of the negative of the Killing form of $\mathfrak{g}$ to the entire group. The Laplacian of G is then the negative of the Casimir operator and the zeta-function of (G, g) is

$$(0) \quad Z(t) = \frac{1}{|w|} \sum_{\Lambda \in L} f^2(\Lambda) \exp[-(|\Lambda|^2 - |\delta|^2)t].$$

In (0), $f(\Lambda) = \Pi_{\alpha > 0}(\Lambda, \alpha) / \Pi_{\alpha > 0}(\delta, \alpha)$, $\delta = \frac{1}{2} \sum_{\alpha > 0} \alpha$, L is the lattice of integral weights, $|w|$ is the order of the Weyl group and inner products and norms are with respect to the Killing form. We will prove the following

THEOREM .

$$Z(t) = \frac{\text{Vol } G}{(4\pi t)^{\dim G/2}} \exp|\delta|^2 t + \text{exponentially small error as } t \downarrow 0.$$

1. Theta relations. In this section we will state some well-known theta relations with minor modifications that are necessary for our purposes. Let $\mathbf{R}^n$ be Euclidean n -space endowed with the usual inner product. Then the Laplacian on $\mathbf{R}^n$, Δ , is just $\partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$ where $\{x_1, \dots, x_n\}$ is an orthonormal basis. We assume $P_q(x)$ is a homogeneous polynomial of degree q such that $\Delta P_q \equiv 0$.

LEMMA 1.

$$(1) \quad \sum_{m \in \mathbf{Z}^n} P_q(m) \exp(-t\pi|m|^2) = i^q t^{-n/2-q} \sum_{m \in \mathbf{Z}^n} P_q(m) \exp(-\pi|m|^2/t).$$

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PROOF. This result is found in [1].

If $P_q \equiv 1$ we obtain the classical theta relation. It is important to remark that if $q \geq 1$ then $P_q(0) = 0$ and the right-hand side of (1) is ES (exponentially small) as $t \downarrow 0$.

Lemma 1 may be modified by taking a general lattice. If L is a lattice and L' is its dual then

LEMMA 2.

$$(2) \quad \sum_{m \in L} P_q(m) \exp(-\pi t |m|^2) = V_L^{-1} i^q t^{-n/2-q} \sum_{m \in L'} P_q(m) \exp(-\pi |m|^2/t)$$

where V_L = volume of a fundamental parallelepiped of L .

PROOF. Lemma 2 follows directly from the Poisson summation formula.

We now proceed to apply Lemma 2 to the problem at hand.

2. Preliminary steps. We now let G be a fixed group and n be the rank of G . We fix a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$, $\mathfrak{r}$, and identify $\mathbf{R}^n$ with the real span of the roots of $\mathfrak{g}_{\mathbb{C}}$ in $\mathfrak{r}$, the complex dual of $\mathfrak{r}$. L will be the lattice of integral weights. The natural inner product on $\mathbf{R}^n$ will be given by the Killing form. With respect to this inner product, we pick an orthonormal basis $\{x_1, \dots, x_n\}$ and, as before, $\Delta = \partial^2/\partial x_1^2 + \dots + \partial^2/\partial x_n^2$. We now consider 2 cases: $n = 1$ and $n > 1$.

3. Rank $G = 1$. If Rank $G = 1$, $\mathfrak{g}$ is A_1 and $f(\Lambda)$ is particularly simple. We may take $f(\Lambda) = \Lambda$ and $L = \mathbf{Z}$. Then

$$Z(t) = \frac{1}{2} \exp|\delta|^2 t \sum_{\Lambda \in \mathbf{Z}} \Lambda^2 \exp(-|\Lambda|^2 t) \quad \text{with } |\Lambda|^2 = 2\Lambda^2.$$

But

$$\begin{aligned} \sum_{\Lambda \in \mathbf{Z}} \Lambda^2 \exp(-2\Lambda^2 t) &= -\frac{d}{dt} \left(\frac{1}{2} \sum_{\Lambda \in \mathbf{Z}} \exp(-2\Lambda^2 t) \right) \\ &= -\frac{d}{dt} \left(\frac{\pi^{1/2}}{2^{3/2} t^{1/2}} \sum_{\Lambda \in \mathbf{Z}} \exp(-\Lambda^2 \pi^2/2t) \right) \\ &= \frac{\pi^{1/2}}{2^{5/2} t^{3/2}} \sum_{\Lambda \in \mathbf{Z}} \exp(-\pi^2 \Lambda^2/t) - \frac{\pi^{5/2}}{2^{5/2} t^{5/2}} \sum_{\Lambda \in \mathbf{Z}} \Lambda^2 \exp(-\Lambda^2 \pi^2/2t) \\ &= \frac{\pi^{1/2}}{2^{5/2} t^{3/2}} + \text{ES} \quad \text{as } t \downarrow 0. \end{aligned}$$

Thus

$$\begin{aligned} Z(t) &= \frac{\pi^{1/2}}{2^{7/2} t^{3/2}} \exp|\delta|^2 t + \text{ES} \quad \text{as } t \downarrow 0 \\ &= \frac{\text{Vol } G}{(4\pi t)^{3/2}} \exp|\delta|^2 t + \text{ES} \quad \text{as } t \downarrow 0 \text{ (by [2])}. \end{aligned}$$

4. Rank $G > 1$. In this case we need one auxiliary result.

LEMMA 3. Let $h_q(x)$ be a homogeneous polynomial of degree q in $\mathbf{R}^n$ with $n > 1$. Then

$$h_q(x) = P_0(x) + |x|^2 P_1(x) + \cdots + |x|^{2d} P_d(x)$$

where $P_i(x)$ is a harmonic homogeneous polynomial of degree $q - 2i$ and $d = [q/2]$.

PROOF. See [3, p. 139].

Applying Lemma 3 to $f^2(x)$ if Rank $G > 1$, we see $f^2(x) = P_0(x) + |x|^2 P_1(x) + \cdots + c|x|^{2a}$ where a is the degree of $f(x)$ which is the number of positive roots of $\mathfrak{g}_{\mathbf{C}}$. $P_d(x) = c$ a polynomial of degree 0. Then

$$\begin{aligned} Z(t) &= \frac{\exp|\delta|^2 t}{|w|} \sum_{\Lambda \in L} f^2(\Lambda) \exp(-|\Lambda|^2 t) \\ &= \frac{\exp|\delta|^2 t}{|w|} \sum_{k=0}^a \sum_{\Lambda \in L} P_k(\Lambda) |\Lambda|^{2k} \exp(-|\Lambda|^2 t). \end{aligned}$$

We will show that if $k < a$ then

$$(3) \quad \sum_{\Lambda \in L} P_k(\Lambda) |\Lambda|^{2k} \exp(-|\Lambda|^2 t)$$

is exponentially small as $t \downarrow 0$. To do so, merely observe (3) equals

$$\begin{aligned} &(-1)^k \frac{d^k}{dt^k} \left(\sum_{\Lambda \in L} P_k(\Lambda) \exp(-|\Lambda|^2 t) \right) \\ &= (-1)^k \frac{d^k}{dt^k} \left(V_L^{-1} i^{2k} (t/\pi)^{-n/2-2k} \sum_{\Lambda \in L'} P_k(\Lambda) \exp(-\pi^2 |\Lambda|^2/t) \right). \end{aligned}$$

After differentiating we have a finite sum of terms of the form

$$c' t^{-r} \sum_{\Lambda \in L'} Q_l(\Lambda) \exp(-\pi^2 |\Lambda|^2/t)$$

with $Q_l(x)$ homogeneous of degree $l \geq 2a - 2k$. Then $Q_l(0) = 0$ and the sum is exponentially small as $t \downarrow 0$. Thus we need only consider the single term

$$(4) \quad \sum_{\Lambda \in L} c |\Lambda|^{2a} \exp(-|\Lambda|^2 t).$$

This equals

$$\begin{aligned} (-1)^a \frac{d^a}{dt^a} \left(c \sum_{\Lambda \in L} \exp(-|\Lambda|^2 t) \right) &= (-1)^a \frac{d^a}{dt^a} \left(\frac{c}{V_L} \frac{\pi^{n/2}}{t^{n/2}} \sum_{\Lambda \in L'} \exp(-|\Lambda|^2 \pi^2/t) \right) \\ &= \frac{c''}{t^{n/2+a}} \sum_{\Lambda \in L'} \exp(-|\Lambda|^2 \pi^2/t) + \text{ES} \quad \text{as } t \downarrow 0 \\ &= \frac{c''}{t^{\dim G/2}} + \text{ES} \quad \text{as } t \downarrow 0. \end{aligned}$$

So

$$Z(t) = \frac{\exp|\delta|^2 t}{|w|} \frac{c''}{t^{\dim G/2}} + \text{ES} \quad \text{as } t \downarrow 0.$$

However $Z(t) \sim (\text{Vol } G)/(4\pi t)^{\dim G/2}$ as $t \downarrow 0$, so

$$Z(t) = \frac{\text{Vol } G}{(4\pi t)^{\dim G/2}} \exp|\delta|^2 t + \text{ES} \quad \text{as } t \downarrow 0.$$

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