INEQUALITIES FOR $\max |S_{\mathbf{k}}|/b_{\mathbf{k}}$ WHERE $\mathbf{k} \in N^r$

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ABSTRACT. Lemma 1 presents a powerful general inequality for $\max |S_{\mathbf{k}}|/b_{\mathbf{k}}$. This is applied in multidimensional time to sums of independent random variables and martingales to yield both old and new inequalities of the Doob, Hájek-Rényi, Skorokhod and Marcinkiewicz-Zygmund types. A brief application is made to the partial sum process.

1. A general inequality. Let N denote the positive integers; for $r \geq 1$ let N^r denote the set of r -tuples $\mathbf{k} = (k_1, \dots, k_r)$, $k_i \in N$. Let $\{X_{\mathbf{k}}\}_{\mathbf{k} \in N^r}$ be random variables. Let $\{b_{\mathbf{k}}\}_{\mathbf{k} \in N^r}$ be a set of positive constants such that

$$(1) \quad \Delta b_{\mathbf{k}} \geq 0 \quad \text{for all } \mathbf{k} \geq \mathbf{1};$$

here $\Delta b_{\mathbf{k}}$ is the usual r -dimensional differencing of $b_{\mathbf{k}}$ around the 2^r points of N^r neighboring $\mathbf{k}$ which are $\leq \mathbf{k}$ in the usual coordinatewise ordering, and $\mathbf{1} = (1, \dots, 1)$. Define

$$(2) \quad S_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} X_{\mathbf{k}}, \quad M_{\mathbf{n}} = \max_{\mathbf{k} \leq \mathbf{n}} |S_{\mathbf{k}}|, \quad Y_{\mathbf{k}} = X_{\mathbf{k}}/b_{\mathbf{k}}, \quad T_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} Y_{\mathbf{k}}.$$

The following lemma is useful for proving inequalities for $\max |S_{\mathbf{k}}|/b_{\mathbf{k}}$.

LEMMA 1. *If the positive $b_{\mathbf{k}}$ satisfy (1), then for arbitrary $X_{\mathbf{k}}$,*

$$\max_{\mathbf{k} \leq \mathbf{n}} |S_{\mathbf{k}}|/b_{\mathbf{k}} \leq 2^r \max_{\mathbf{k} \leq \mathbf{n}} |T_{\mathbf{k}}|.$$

If all $T_{\mathbf{k}}$ are ≥ 0 , we may replace 2^r by 2^{r-1} .

PROOF. Define $b_{\mathbf{k}}$ and $X_{\mathbf{k}}$ to be zero if some $k_i = 0$, $1 \leq i \leq r$. Then (cf. [13]):

$$S_{\mathbf{k}} = \sum_{\mathbf{j} \leq \mathbf{k}} b_{\mathbf{j}} (\Delta T_{\mathbf{j}}) = \sum_{\mathbf{j} \leq \mathbf{k}} (\Delta T_{\mathbf{j}}) \sum_{\mathbf{i} \leq \mathbf{j}} (\Delta b_{\mathbf{i}}) = \sum_{\mathbf{i} \leq \mathbf{k}} T_{\mathbf{ik}} (\Delta b_{\mathbf{i}})$$

where $T_{\mathbf{ik}} = \sum_{\mathbf{i} \leq \mathbf{j} \leq \mathbf{k}} Y_{\mathbf{j}}$. Note that each $T_{\mathbf{ik}}$ is formed by differencing 2^r sums of the type $T_{\mathbf{j}}$. Since $\sum_{\mathbf{i} \leq \mathbf{k}} (\Delta b_{\mathbf{i}}/b_{\mathbf{k}}) = 1$ with each $\Delta b_{\mathbf{i}} \geq 0$ we thus have

$$\begin{aligned} \max_{\mathbf{k} \leq \mathbf{n}} |S_{\mathbf{k}}|/b_{\mathbf{k}} &\leq \max_{\mathbf{k} \leq \mathbf{n}} \sum_{\mathbf{i} \leq \mathbf{k}} (\Delta b_{\mathbf{i}}) |T_{\mathbf{ik}}|/b_{\mathbf{k}} \\ &\leq 1 \max_{\mathbf{k} \leq \mathbf{n}} \max_{\mathbf{i} \leq \mathbf{k}} |T_{\mathbf{ik}}| \leq 2^r \max_{\mathbf{k} \leq \mathbf{n}} |T_{\mathbf{k}}|. \end{aligned}$$

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Clearly 2^{r-1} works if $T_{\mathbf{k}} \geq 0$ for all $\mathbf{k}$. $\square$

In the next section we apply Lemma 1 to deduce results concerning $\max |S_{\mathbf{k}}|/b_{\mathbf{k}}$ from both known and new inequalities for $M_{\mathbf{n}}$.

2. Martingale inequalities in N^r . Let $\{\mathfrak{F}_{\mathbf{k}}\}_{\mathbf{k} \in N^r}$ be a collection of σ -fields such that $\mathbf{k} \leq \mathbf{n}$ implies $\mathfrak{F}_{\mathbf{k}} \subset \mathfrak{F}_{\mathbf{n}}$ and let $Z_{\mathbf{k}}$ be $\mathfrak{F}_{\mathbf{k}}$ -integrable. We say that $(Z_{\mathbf{k}}, \mathfrak{F}_{\mathbf{k}})$ is a martingale* (MG*) (resp., submartingale* (SMG*)) if

$$(3) \quad E(Z_{\mathbf{k}}|\mathfrak{F}_{\mathbf{i}}) = (\text{resp., } \geq) Z_{\mathbf{i} \wedge \mathbf{k}} \quad \text{for all } \mathbf{i}, \mathbf{k}.$$

When $r = 1$ this is the usual definition of a MG or SMG, but for $r > 1$ it is stronger. Let $S_{\mathbf{k}}$ be as in (2); note that

- (i) if $(S_{\mathbf{k}}, \mathfrak{F}_{\mathbf{k}})$ is a MG*, so is $(T_{\mathbf{k}}, \mathfrak{F}_{\mathbf{k}})$;
- (ii) if $\hat{\mathbf{k}} = (k_1, \dots, k_{r-1})$ and $L_{\hat{\mathbf{k}}} \equiv \max_{k_r \leq n_r} |S_{(\hat{\mathbf{k}}, k_r)}|$, then $(L_{\hat{\mathbf{k}}}, \mathfrak{F}_{(\hat{\mathbf{k}}, n_r)})$ is an $r - 1$ dimensional SMG*, provided that $(S_{\mathbf{k}}, \mathfrak{F}_{\mathbf{k}})$ is a MG* or nonnegative SMG*.

To show (ii), note that $|S_{\mathbf{k}}|$ is a SMG*; and thus

$$\begin{aligned} E(L_{\hat{\mathbf{k}}}|\mathfrak{F}_{(\hat{\mathbf{k}}, n_r)}) &\geq \max_{k_r \leq n_r} E(|S_{(\hat{\mathbf{k}}, k_r)}||\mathfrak{F}_{(\hat{\mathbf{k}}, n_r)}) \\ &\geq \max_{k_r \leq n_r} |S_{(\hat{\mathbf{k}} \wedge \mathbf{i}, k_r)}| = L_{\hat{\mathbf{k}} \wedge \mathbf{i}}. \end{aligned}$$

We assume henceforth that $(S_{\mathbf{k}}, \mathfrak{F}_{\mathbf{k}})$ is a MG* or nonnegative SMG*. An inductive proof based on (ii) easily yields

$$(4) \quad E\{M_{\mathbf{n}}^p\} \leq (p/(p-1))^p E|S_{\mathbf{n}}|^p, \quad p > 1, r \geq 1.$$

This is due to Doob (1953, p. 317) when $r = 1$ and to Cairoli (1970) when $r > 1$ (see Wichura (1969) in case of independence and $p = 2$). Another inductive argument based on Doob's upcrossing inequality leads to the new inequality

$$\begin{aligned} (5) \quad P\{M_{\mathbf{n}} \geq 2^r \lambda\} &\leq \lambda^{-1} \int_{\lambda}^{\infty} (\log u)^{r-1} P\{|S_{\mathbf{n}}| \geq u\} du \\ &\leq \lambda^{-1} \int_{\{|S_{\mathbf{n}}| \geq \lambda\}} |S_{\mathbf{n}}| (\log |S_{\mathbf{n}}|)^{r-1} dP, \quad r \geq 1, \lambda \geq 1. \end{aligned}$$

is a Skorokhod-type improvement of another of Cairoli's inequalities [5, p. 3]; see §4 for the proof of (5).

In this paragraph suppose the $X_{\mathbf{k}}$ are independent with zero means. Additional inequalities are then possible. Wichura (1969) has proved the following Skorokhod-type inequality:

$$(6) \quad \begin{aligned} P\{M_{\mathbf{n}} \geq 2^r \lambda\} &\leq (1 - \text{Var}[S_{\mathbf{n}}]/\lambda^2)^{-r} P\{|S_{\mathbf{n}}| \geq \lambda\}, \\ r &\geq 1, \lambda^2 \geq \text{Var}[S_{\mathbf{n}}]. \end{aligned}$$

A very interesting inequality involving only the first moment is

$$(7) \quad P\{M_{\mathbf{n}} \geq \lambda\} \leq \lambda^{-1} E\{M_{\mathbf{n}}\} \leq \lambda^{-1} (2^{r+1} + 1) E|S_{\mathbf{n}}|, \quad r \geq 1, \lambda > 0.$$

For the case $r = 1$, see Hunt (1966, p. 69); for $r = 2$ the result has been

established independently by Gabriel (1974). A proof of (7) is given in §4.

By applying Lemma 1 in conjunction with the above inequalities, some useful inequalities for $\max |S_k|/b_k$ can be derived. Applying Chebyshev's inequality along with Lemma 1 and (4) we get

$$\begin{aligned} P \left\{ \max_{k \leq n} |S_k|/b_k \geq \lambda \right\} &\leq \lambda^{-p} 2^{rp} E \left(\max_{k \leq n} |T_k|^p \right) \\ (8) \quad &\leq \lambda^{-p} 2^{rp} (p/(p-1))^p E(|T_n|^p), \quad r \geq 1, \lambda > 0, p > 1, \\ &= \lambda^{-2} 4^{2r} \sum_{k \leq n} \text{Var} [X_k] b_k^2 \quad \text{when } p = 2 \end{aligned}$$

for any MG* (cf. Smythe (1974)).

Now suppose in this paragraph that the X_k are independent with zero means. Application of Lemma 1 to (6) gives

$$\begin{aligned} (9) \quad P \left\{ \max_{k \leq n} |S_k|/b_k \geq 4^r \lambda \right\} &\leq (1 - \text{Var} [T_n]/\lambda^2)^{-r} P\{|T_n| \geq \lambda\}, \\ &r \geq 1, \lambda^2 \geq \text{Var} [T_n]. \end{aligned}$$

This inequality seems to be new, even for $r = 1$. Application of Lemma 1 to (7) gives

$$\begin{aligned} (10) \quad P \left\{ \max_{k \leq n} |S_k|/b_k \geq \lambda \right\} &\leq \lambda^{-1} 2^r E \left\{ \max_{k \leq n} |T_k| \right\} \\ &\leq \lambda^{-1} 2^r (2^{r+1} + 1) E|T_n|, \quad r \geq 1, \lambda > 0. \end{aligned}$$

Combining Lemma 1 with (8) and (10) we obtain

$$\begin{aligned} (11) \quad E \left(\max_{k \leq n} |S_k|^p / b_k^p \right) &\leq C_{p,r} E|T_n|^p \leq 2C_{p,r} \sum_{k \leq n} E|X_k|^p b_k^p, \\ &r \geq 1, 1 \leq p \leq 2; \end{aligned}$$

where the second inequality for $1 \leq p \leq 2$ follows from von Bahr and Esseen (1965). The constant $C_{p,r}$ equals $2^r(2^{r+1} + 1)$ for $p = 1$ and equals $(2p/(p-1))^p$ for $1 < p \leq 2$. Equation (11) generalizes some results for $r = 1$ due to Bickel (1970) and Dunnage (1972).

In closing this section, we remark first that for $r = 1$, the Birnbaum-Marshall-Chow inequality [3], [6] can be strengthened along the lines of (5), and second that Lemma 1 can be applied to various continuous parameter processes by a passage to the limit.

3. Applications. We indicate here some applications of the results of §2.

(A) Let $\mathbf{t} = (t_1, \dots, t_r)$, $I \equiv \{\mathbf{t}: 0 \leq \mathbf{t} \leq 1\}$. Define

$$(12) \quad Z_n(\mathbf{t}) = \sum_{k \leq n\mathbf{t}} X_k / (\sigma^2 n_1 \cdots n_r)^{1/2}$$

where the X_k are i.i.d. with mean 0, variance σ^2 , and $n\mathbf{t} = (n_1 t_1, \dots, n_r t_r)$. Let q be a continuous function defined on I such that

$$(13) \quad \int_0^1 [q(\mathbf{t})]^{-2} \prod_1^r dt_i < \infty \quad \text{and} \quad \Delta q \geq 0,$$

where Δq denotes the obvious analog of (1). Let ρ denote the supremum metric, and let

$$\rho_q(f, g) \equiv \sup_{t \in I} |f(t) - g(t)|/q(t).$$

Let D denote the set of all functions on I which are "continuous from above with limits from below", as made precise in [2]. Let Z denote the Wiener process in r -dimensional time (see [16], for example). Then

$$(14) \quad Z_n \Rightarrow Z \text{ on } (D, \rho_q) \quad \text{as } n \rightarrow \infty;$$

where " $\Rightarrow$ " means that $E[\psi(Z_n)] \rightarrow E[\psi(Z)]$ as $n \rightarrow \infty$ for all bounded functions ψ which are ρ_q -continuous on D and are measurable with respect to the σ -field generated by the finite-dimensional subsets of D . Our proof will clearly carry over to the non-i.i.d case considered in [2], the only problem being added notation.

PROOF OF (14). $Z_n \Rightarrow Z$ on (D, ρ) is known from [2]; and as in [12], we may suppose $\rho(Z_n, Z) \xrightarrow{P} 0$ by using different versions of the processes. For a real θ to be specified below, let $\theta_i = (1, 1, \dots, 1, \theta, 1, \dots, 1)$ for $1 \leq i \leq r$. Let $b_k = q(k/n)$, where $k/n = (k_1/n_1, \dots, k_r/n_r)$ and n is fixed for the moment. Then for any $\delta > 0$ we have from (8) that

$$\begin{aligned} P \left[\sup_{0 \leq t \leq \theta_i} \frac{|Z_n(t)|}{q(t)} \geq \delta \right] &\leq P \left[\max_{k \leq n\theta_i} \frac{|S_k|}{b_k} \geq \delta \sigma \sqrt{n_1 \cdots n_r / 2} \right] \\ &\leq \left(\frac{4^{2r+1}}{\delta^2 n_1 \cdots n_r} \right) \sum_{k \leq n\theta_i} b_k^{-2} \\ &\leq \left(\frac{4^{2r+1}}{\delta^2} \right) \int_0^{\theta_i} [q(t)]^{-2} \prod_1^r dt_i. \end{aligned}$$

From (13) we may choose θ to make the last term above $< \delta$. Using a limiting argument based on the continuity of the Brownian process, it is clear that the same inequality can be derived for Z . Since

$$\rho_q(Z_n, Z) \leq \sum_{i=1}^r \sup_{0 \leq t \leq \theta_i} \frac{|Z_n(t)|}{q(t)} + \sum_{i=1}^r \sup_{0 \leq t \leq \theta_i} \frac{|Z(t)|}{q(t)} + \frac{\rho(Z_n, Z)}{q(\theta \mathbf{1})},$$

we have

$$P(\rho_q(Z_n, Z) \geq (2r+1)\delta) \leq 2r\delta + P(\rho(Z_n, Z) \geq \delta q(\theta \mathbf{1})) \leq (2r+1)\delta$$

for n exceeding some n_δ . Thus $\rho_q(Z_n, Z) \xrightarrow{P} 0$; hence for the original processes, $Z_n \Rightarrow Z$ on (D, ρ_q) . (Note the proof of Theorem 2.1 of [12].) $\square$

(B) One use of (6) is in establishing the tightness of the partial sum process, where one uses the asymptotic normality to suitably approximate $P(|S_k| \geq \lambda)$. The same technique could be applied to (5) (cf. [4] for $r = 1$); the integral of $|S_k|(\log |S_k|)^{r-1}$ over $[\lambda, \infty)$ may be approximated by its limiting normal value, because it is dominated by S_n^2 .

(C) Useful results for empirical processes based on Lemma 1 and (5) have

been obtained in a Ph.D. thesis by Jon Wellner of the University of Washington.

(D) Inequality (8) has been used to establish strong laws of large numbers for i.i.d. random variables [13]; inequality (11) is used in [14] to establish a law of large numbers of the form $P\{|S_k|/b_k > \epsilon \text{ i.o.}\} = 0$ for nonidentically distributed X_k .

4. Proofs of (5) and (7).

PROOF OF (5). Since $\{M_n \geq 2\lambda\}$ implies, taking $S_0 \equiv 0$, that either an upcrossing of $[\lambda, 2\lambda]$ or a downcrossing of $[-\lambda, -2\lambda]$ takes place, Doob's upcrossing inequality [7, p. 316] yields (see also Brown (1971))

$$\begin{aligned} P\{M_n \geq 2\lambda\} &\leq \lambda^{-1} \int_{\{|S_n| \geq \lambda\}} (|S_n| - \lambda) dP \\ (15) \qquad &= \lambda^{-1} \int_{\lambda}^{\infty} P(|S_n| \geq t) dt, \quad r = 1, \lambda > 0. \end{aligned}$$

Now note that (ii) of §2 extends easily to show that $L_k \equiv \max_{1 \leq \hat{\mathbf{i}} \leq \hat{\mathbf{k}} \leq \hat{\mathbf{n}}} |S_{\hat{\mathbf{k}}, k}|$, $1 \leq k \leq n_r$, is a one-dimensional SMG with respect to the σ -fields $\mathfrak{F}_{\hat{\mathbf{n}}, k}$. Assuming the validity of (5) in dimension $r - 1$,

$$\begin{aligned} P\{M_{\mathbf{n}} \geq 2^r \lambda\} &= P\left(\max_{1 \leq k \leq n_r} L_k \geq 2^r \lambda\right) \\ &\leq (2^{r-1} \lambda)^{-1} \int_{2^{r-1} \lambda}^{\infty} P(L_{n_r} \geq t) dt \quad (\text{by (15)}) \\ &= \lambda^{-1} \int_{\lambda}^{\infty} P(L_{n_r} \geq 2^{r-1} s) ds \\ &\leq \lambda^{-1} \int_{\lambda}^{\infty} s^{-1} \int_s^{\infty} (\log^{r-2} u) P\{|S_{\mathbf{n}}| \geq u\} du ds \\ &= \lambda^{-1} \int_{\lambda}^{\infty} (\log^{r-1} u) P\{|S_{\mathbf{n}}| \geq u\} du \\ &\quad - \frac{\ln \lambda}{\lambda} \int_{\lambda}^{\infty} (\log^{r-2} u) P\{|S_{\mathbf{n}}| \geq u\} du \\ &\leq \lambda^{-1} \int_{\lambda}^{\infty} (\log^{r-1} u) P\{|S_{\mathbf{n}}| \geq u\} du \quad \text{for } \lambda \geq 1. \end{aligned}$$

The second inequality of (5) is easily verified by induction, using the relation

$$\begin{aligned} \int_{\{Z \geq \lambda\}} Z (\log Z)^r dP &= \lambda (\log \lambda)^r P(Z \geq \lambda) \\ &\quad + \int_{\lambda}^{\infty} P(Z \geq t) \frac{d}{dt} [t (\log t)^r] dt, \quad Z \geq 0, \lambda > 0. \end{aligned}$$

PROOF OF (7). The proof rests upon the following result (see, e.g., [11, Theorem 1]): If the X_k are independent and symmetric, then

$$(16) \quad P\left\{\sup_{\mathbf{k} \leq \mathbf{n}} |S_{\mathbf{k}}| \geq \lambda\right\} \leq 2^r P\{|S_{\mathbf{n}}| \geq \lambda\}, \quad r \geq 1, \lambda > 0.$$

One next observes that for symmetric X_k , using (16),

$$(17) \quad E\{M_n\} = \int_0^\infty P\{M_n \geq \lambda\} d\lambda \leq 2^r \int_0^\infty P\{|S_n| \geq \lambda\} d\lambda \leq 2^r E|S_n|.$$

For X_k not necessarily symmetric, one symmetrizes the X_k , applies (16) and (17), and then “desymmetrizes”, as in Hunt (1966), to give the result in (7). $\square$

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