ASYMPTOTIC STABILITY FOR ABSTRACT NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. The nonlinear autonomous functional differential equation $\dot{x}(t) = f(x(t)) + g(x_t)$, $t \geq 0$, $x_0 = \phi$ is investigated by means of the theory of semigroups of nonlinear operators. The properties of the semigroup associated with this equation provide stability information about the solutions.

1. Introduction. The purpose of this paper is to prove some stability properties of the nonlinear autonomous functional differential equation

$$(FDE) \quad \dot{x}(\phi)(t) = f(x(\phi)(t)) + g(x_t(\phi)), \quad t \geq 0, x_0(\phi) = \phi.$$

The notation of (FDE) follows J. Hale [4], that is $\phi \in C = C([-r, 0]; X)$ where $r > 0$ and X is a Banach space, $x(\phi)(t): [-r, \infty) \rightarrow X$, and $x_t(\phi) \in C$ is defined for each $t \geq 0$ by $x_t(\phi)(\theta) = x(\phi)(t + \theta)$, $\theta \in [-r, 0]$. In (FDE) we will require a Lipschitz condition on the nonlinear operator $g: C \rightarrow X$ and an accretiveness condition on the nonlinear operator $f: X \rightarrow X$. The ordinary part of (FDE) corresponding to f will act as a damping term for the equation. Our main result can be summarized as follows: Suppose g has Lipschitz constant β and $-f + \alpha I$ is accretive. If $\alpha = -\beta$, then (FDE) is stable, and if $\alpha < -\beta$, then (FDE) is asymptotically stable. As a simple example for our problem one can let $X = \mathbf{R}$, $f(x) = -x^3 + \alpha x$, $g(\phi) = h(\phi(-r))$, where h has Lipschitz constant β , and then (FDE) is the scalar delay equation

$$\dot{x}(\phi)(t) = -x^3(\phi)(t) + \alpha x(\phi)(t) + h(x(\phi)(t - r)).$$

Our approach will be to use the general theory of semigroups of nonlinear operators. By allowing X to be a Banach space our results may be applied to partial functional differential equations as in [7]. For related treatments of our problem one should see [7]–[9].

2. Definitions. For an arbitrary Banach space Y , a nonlinear operator A from Y to Y is *accretive* provided

$$(2.1) \quad \|(I + \lambda A)x - (I + \lambda A)y\| \geq \|x - y\| \quad \text{for all } x, y \in D(A), \lambda > 0.$$

Our results rely upon the following general theorem, due to M. Crandall and T. Liggett [1], from nonlinear semigroup theory: Suppose for some $\gamma \in \mathbf{R}$,

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$A + \gamma I: Y \rightarrow Y$ is accretive and $R(I + \lambda A)$ is onto for all sufficiently small $\lambda > 0$. Then,

$$(2.2) \quad \lim_{n \rightarrow \infty} (I + t/nA)^{-n} x \stackrel{\text{def}}{=} T(t)x \text{ exists for } x \in \overline{D(A)}, t \geq 0.$$

Moreover, $T(t)$, $t \geq 0$, is a *strongly continuous semigroup of nonlinear operators* on $\overline{D(A)}$, that is,

$$(2.3) \quad T(0)x = x \quad \text{for all } x \in \overline{D(A)};$$

$$(2.4) \quad T(t)x \text{ is continuous in } t \text{ for each fixed } x \in \overline{D(A)};$$

$$(2.5) \quad T(t_1 + t_2) = T(t_1)T(t_2) \quad \text{for } t_1, t_2 \geq 0;$$

$$(2.6) \quad \|T(t)x - T(t)y\| \leq e^{\gamma t} \|x - y\| \quad \text{for } x, y \in \overline{D(A)}, t \geq 0.$$

Let C_0 be the subspace of C given by $C_0 = \{\phi \in C: \phi(0) = 0\}$. Define the linear operators

$$(2.7) \quad A_0: C_0 \rightarrow C_0 \quad \text{by } A_0\phi = -\phi', \quad D(A_0) = \{\phi \in C_0: \phi' \in C_0\},$$

$$(2.8) \quad A_1: C \rightarrow C \quad \text{by } A_1\phi = -\phi', \quad D(A_1) = \{\phi \in C_0: \phi' \in C\}.$$

From the theory of semigroups of linear operators one has that A_0 is accretive in C_0 , $R(I + \lambda A_0) = C_0$ for $\lambda > 0$, $D(A_0)$ is dense in C_0 , A_0 is the infinitesimal generator of a strongly continuous semigroup of linear contractions on C_0 , and

$$(2.9) \quad \lim_{\lambda \rightarrow 0^+} (I + \lambda A_0)^{-1} \phi = \phi \quad \text{for all } \phi \in C_0.$$

Also, A_1 is accretive in C , $R(I + \lambda A_1) = C$ for $\lambda > 0$, but $D(A_1)$ is not dense in C . If $\lambda > 0$ then

$$(2.10) \quad ((I + \lambda A_0)^{-1} \phi)(\theta) = \left(\frac{e^{\theta/\lambda}}{\lambda} \right) \int_{\theta}^0 e^{-s/\lambda} \phi(s) ds, \quad \phi \in C_0, \theta \in [-r, 0];$$

$$(2.11) \quad ((I + \lambda A_1)^{-1} \phi)(\theta) = \left(\frac{e^{\theta/\lambda}}{\lambda} \right) \int_{\theta}^0 e^{-s/\lambda} \phi(s) ds, \quad \phi \in C, \theta \in [-r, 0];$$

$$(2.12) \quad (I + \lambda A_1)^{-1} \phi = (I + \lambda A_0)^{-1}(\phi - \phi(0)) + (1 - e^{\theta/\lambda})\phi(0), \quad \phi \in C.$$

3. The nonlinear semigroup for (FDE). In this section we shall construct a nonlinear semigroup which can be associated with (FDE) by exhibiting its generator in the sense of (2.2). In what follows we shall suppose that for some $\alpha \in \mathbf{R}$, $f: X \rightarrow X$ such that $-f + \alpha I$ is accretive and $R(I - \lambda f) = X$ for $0 < \lambda < 1/\max\{0, \alpha\}$. From (2.1) this implies that for $x, y \in D(f)$, $\lambda > 0$,

$$(3.1) \quad \|(I - \lambda f)^{-1}x - (I - \lambda f)^{-1}y\| \leq (1/(1 - \lambda\alpha)) \|x - y\|,$$

and

$$(3.2) \quad \lim_{\lambda \rightarrow 0^+} (I - \lambda f)^{-1} x = x \quad \text{for } x \in \overline{D(f)} \quad (\text{see [1, Lemma 1.2(ii)]}).$$

We shall also suppose that $g: C \rightarrow X$ is Lipschitz continuous with Lipschitz constant β . Define $A: C \rightarrow C$ by $A\phi = -\phi'$ with $D(A) = \{\phi \in C: \phi' \in C, \phi(0) \in D(f), \phi'(0) = f(\phi(0)) + g(\phi)\}$.

PROPOSITION 1. $A + \gamma I$ is accretive in C and $R(I + \lambda A) = C$ for $0 < \lambda < 1/\gamma$, where $\gamma = \max\{0, \alpha + \beta\}$.

PROOF. We first show $I + \lambda A$ is 1-1 and onto. Let $0 < \lambda < 1/\gamma$, $\psi \in C$. Define the mapping $k: X \rightarrow X$ by

$$(3.3) \quad k(b) = (I - \lambda f)^{-1}(\psi(0) + \lambda g(e^{\theta/\lambda} b + (I + \lambda A_1)^{-1} \psi)), \quad b \in X.$$

Then k is a strict contraction on X , since $\|k(b_1) - k(b_2)\| \leq (\lambda\beta/(1 - \lambda\alpha)) \cdot \|b_1 - b_2\|$. Thus, k has a unique fixed point b_0 , and

$$(3.4) \quad \phi(\theta) \stackrel{\text{def}}{=} e^{\theta/\lambda} b_0 + ((I + \lambda A_1)^{-1} \psi)(\theta)$$

solves uniquely $\phi - \lambda\phi' = \psi$, $\phi'(0) = f(\phi(0)) + g(\phi)$, that is, $(I + \lambda A)\phi = \psi$.

Next, we will show that for all $\psi_1, \psi_2 \in C$, $0 < \lambda < 1/\gamma$,

$$(3.5) \quad \|(I + \lambda A)^{-1} \psi_1 - (I + \lambda A)^{-1} \psi_2\| \leq (1/(1 - \lambda\gamma)) \|\psi_1 - \psi_2\|.$$

Let $(I + \lambda A)\phi_1 = \psi_1$, $(I + \lambda A)\phi_2 = \psi_2$, and $\theta \in [-r, 0]$ such that $\|\phi_1(\theta) - \phi_2(\theta)\| = \|\phi_1 - \phi_2\|$. From (3.3), (3.4), and (2.11) we have

$$\begin{aligned} \|\phi_1 - \phi_2\| &= \|\phi_1(\theta) - \phi_2(\theta)\| \\ &\leq (e^{\theta/\lambda}/(1 - \lambda\alpha)) (\|\psi_1(0) - \psi_2(0)\| + \lambda\beta\|\phi_1 - \phi_2\|) \\ &\quad + (1 - e^{\theta/\lambda}) \|\psi_1 - \psi_2\| \end{aligned}$$

which implies

$$\|\phi_1 - \phi_2\| \leq ((1 - \lambda\alpha + \lambda\alpha e^{\theta/\lambda})/(1 - \lambda\alpha - \lambda\beta e^{\theta/\lambda})) \|\psi_1 - \psi_2\|.$$

If $0 \leq \alpha + \beta$, then

$$(1 - \lambda\alpha + \lambda\alpha e^{\theta/\lambda})/(1 - \lambda\alpha - \lambda\beta e^{\theta/\lambda}) \leq 1/(1 - \lambda(\alpha + \beta)) = 1/(1 - \lambda\gamma).$$

If $\alpha + \beta \leq 0$, then

$$\begin{aligned} (1 - \lambda\alpha + \lambda\alpha e^{\theta/\lambda})/(1 - \lambda\alpha - \lambda\beta e^{\theta/\lambda}) \\ \leq (1 - \lambda\alpha - \lambda\beta e^{\theta/\lambda})/(1 - \lambda\alpha - \lambda\beta e^{\theta/\lambda}) = 1/(1 - \lambda\gamma). \end{aligned}$$

In either case, (3.5) holds, and this yields the accretiveness of $A + \gamma I$.

PROPOSITION 2. $\overline{D(A)} = \{\psi \in C: \psi(0) \in \overline{D(f)}\}$.

PROOF. Let $(I + \lambda A)\phi = \psi$ as in Proposition 1. Using (3.3) and (3.4) we see

that

$$\begin{aligned}
 \|\phi(0) - \psi(0)\| &= \|(I - \lambda f)^{-1}(\psi(0) + \lambda g(\phi)) - \psi(0)\| \\
 (3.6) \qquad &\leq (\lambda/(1 - \lambda\alpha))(\beta\|\phi - \psi\| + \|g(\psi)\|) \\
 &\quad + \|(I - \lambda f)^{-1}\psi(0) - \psi(0)\|.
 \end{aligned}$$

Also, from (2.12),

$$(3.7) \quad \|\phi - \psi\| \leq \|(I - \lambda A_0)^{-1}(\psi - \psi(0)) - (\psi - \psi(0))\| + \|\phi(0) - \psi(0)\|.$$

From (3.6) and (3.7) we obtain

$$\begin{aligned}
 \|\phi - \psi\| &\leq ((1 - \lambda\alpha)/(1 - \lambda(\alpha + \beta))) \\
 (3.8) \qquad &\cdot (\|(I - \lambda A_0)^{-1}(\psi - \psi(0)) - (\psi - \psi(0))\| \\
 &\quad + \|(I - \lambda f)^{-1}\psi(0) - \psi(0)\| + (\lambda/(1 - \lambda\alpha))\|g(\psi)\|).
 \end{aligned}$$

From the general theory of accretive operators (see [1, Lemma 1.2 (ii)]) we have from Proposition 1 that

$$(3.9) \quad \overline{D(A)} = \left\{ \psi \in C: \lim_{\lambda \rightarrow 0^+} (I + \lambda A)^{-1}\psi = \psi \right\}.$$

Then, (2.9), (3.2), (3.8), and (3.9) imply the conclusion.

By virtue of Propositions 1 and 2 we may use formula (2.2) to define a nonlinear semigroup $T(t)$, $t \geq 0$, on $\overline{D(A)}$ with generator A . If $\overline{D(f)} = X$, then $\overline{D(A)} = C$. If f is linear and densely defined in X , then $-A$ is exactly the infinitesimal generator of $T(t)$, $t \geq 0$ (see [7] or [8]).

The question arises as to when the semigroup $T(t)$, $t \geq 0$, gives solutions to (FDE). One can use the methods of H. Flaschka and M. Leitman [3] to show that $T(t)$, $t \geq 0$, always has the following property: If for each $\phi \in \overline{D(A)}$ we define

$$\begin{aligned}
 x(\phi)(t) &= \phi(t) \text{ for } -r \leq t \leq 0, \\
 (3.10) \qquad &= (T(t)\phi)(0) \text{ for } t \geq 0;
 \end{aligned}$$

then $T(t)\phi = x_t(\phi)$. In the case that f is everywhere defined and continuous the methods of [3] can be used to show that the function $x(\phi)(t)$ in (3.10) satisfies (FDE) for all $\phi \in D(A)$. The proof of these facts carries over essentially without change from [3]. In the case that X is a Hilbert space we can show the following.

PROPOSITION 3. *If X is a Hilbert space, then for all $\phi \in D(A)$ the function $x(\phi)(t)$ in (3.10) satisfies*

$$(3.11) \quad \dot{x}(\phi)(t) = f(x(\phi)(t)) + g(x_t(\phi)) \quad \text{for a.a. } t \geq 0, \quad x_0(\phi) = \phi.$$

PROOF. We will use the notation and results from [1]. Let $\phi \in D(A)$. We show first that for all $h \in X$, $t \geq 0$,

$$(3.12) \quad \langle (T(t)\phi)(0), h \rangle = \langle \phi(0), h \rangle + \int_0^t \langle (f(T(s)\phi)(0)) + g(T(s)\phi) \rangle ds, h \rangle.$$

As in [3] we have that

$$(T^\lambda(t)\phi)(0) = \phi(0) + \int_0^t (f((J_\lambda T^\lambda(s)\phi)(0)) + g(J_\lambda T^\lambda(s)\phi)) ds,$$

and $J_\lambda T^\lambda(s)\phi$ converges strongly to $T(s)\phi$ as $\lambda \rightarrow 0$. By virtue of the Lebesgue dominated convergence theorem, to establish (3.12) it suffices to show that $f((J_\lambda T^\lambda(s)\phi)(0))$ is bounded in λ and converges weakly to $f((T(s)\phi)(0))$ as $\lambda \rightarrow 0$. The boundedness follows from the Lipschitz continuity of g , the fact that $\phi \in D(A)$, and the inequality

$$\begin{aligned} \text{const} \|A\phi\| &\geq \|AJ_\lambda T^\lambda(s)\phi\| \geq \|(AJ_\lambda T^\lambda(s)\phi)(0)\| \\ &= \|f((J_\lambda T^\lambda(s)\phi)(0)) + g(J_\lambda T^\lambda(s)\phi)\|. \end{aligned}$$

The weak convergence follows from the fact that f is maximal accretive in X by a well-known argument of accretive operator theory (see [2] or [6]). Then (3.12) implies that $x(\phi)(t)$ is weakly differentiable for almost all $t \geq 0$. Also, since $\phi \in D(A)$, $T(t)\phi$ is Lipschitz continuous in t (see (1.11) of [1]), and therefore $x(\phi)(t)$ is strongly of bounded variation in t . By [5, Theorem 3.8.6, p. 88], $x(\phi)(t)$ is strongly differentiable almost everywhere in t and (3.11) holds.

Now we consider the stability properties of (FDE). If $\alpha = -\beta$, then the trajectories of $T(t)$, $t \geq 0$, are stable in the sense of (2.6) with $\gamma = 0$. If $\alpha < -\beta$, the proposition below yields the asymptotic stability of (FDE) if X is a Hilbert space.

PROPOSITION 4. *Suppose $\alpha < -\beta$ and X is a Hilbert space. Then $T(t)$, $t \geq 0$, is asymptotically stable in the sense that there exists a unique point $\psi_0 \in C$ such that $\lim_{t \rightarrow \infty} T(t)\psi = \psi_0$ for all $\psi \in C$. Moreover, ψ_0 is a constant function in C .*

PROOF. Define $j: X \rightarrow X$ by $j(b) = (I - f)^{-1}(b + g(b \cdot \mathbf{1}))$, where $b \in X$ and $\mathbf{1}$ denotes the constant function identically 1 on $[-r, 0]$. Since $\alpha + \beta < 0$, j is a strict contraction with Lipschitz constant $\leq (1 + \beta)/(1 - \alpha)$. Let b_0 be the unique fixed point of j in X and define $\psi_0 = b_0 \cdot \mathbf{1}$. Then $\psi_0 \in D(A)$ and $A\psi_0 = 0$. From formula (2.2) we have that $T(t)\psi_0 = \psi_0$ for all $t \geq 0$.

To prove the conclusion it suffices to show that

$$(3.13) \quad \lim_{t \rightarrow \infty} \sup_{\phi, \psi \in C} \|T(t)\phi - T(t)\psi\| / \|\phi - \psi\| = 0.$$

Let $\phi, \psi \in D(A)$ and let $x(t)$ and $y(t)$ solve (3.11) for ϕ and ψ , respectively. Then, for almost all $t \geq 0$,

$$\begin{aligned} \frac{1}{2}(d/dt)\|x(t) - y(t)\|^2 &= \langle f(x(t)) + g(x_t) - f(y(t)) - g(y_t), x(t) - y(t) \rangle \\ (3.14) \quad &\leq \alpha \|x(t) - y(t)\|^2 + \beta \|T(t)\phi - T(t)\psi\| \|x(t) - y(t)\| \\ &\leq \alpha \|x(t) - y(t)\|^2 + \beta \|\phi - \psi\|^2. \end{aligned}$$

By Gronwall's inequality, (3.14) implies that for $t \geq 0$,

$$(3.15) \quad \|x(t) - y(t)\| \leq \|\phi - \psi\|(\beta/\alpha - e^{2\alpha t}(1 + \beta/\alpha))^{1/2}.$$

Then (3.15) implies that for $t \geq 0$,

$$(3.16) \quad \|T(t)\phi - T(t)\psi\| \leq \|\phi - \psi\|(\beta/\alpha - e^{2\alpha(t-r)}(1 + \beta/\alpha))^{1/2}.$$

Using (2.6) with $\gamma = 0$, we have that (3.16) holds for all $\phi, \psi \in \overline{D(A)}$. But this means that $T(t)$ is a strict contraction when $t > r$ and this fact, together with (2.5), yields (3.13).

In conclusion we remark that all of our propositions carry over easily to the case that f is a multivalued accretive operator, a class of nonlinear operators that is very general and extensively developed (see, e.g., [1] and [2]). Also, Propositions 3 and 4 carry over readily to the case that X is a uniformly convex Banach space.

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