A UNIFORMLY CONVEX BANACH SPACE WITH A SCHAUDER BASIS WHICH IS SUBSYMMETRIC BUT NOT SYMMETRIC

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ABSTRACT. An example of a uniformly convex Banach space with a basis (x_i) is constructed such that (x_i) is subsymmetric but not symmetric.

1. Introduction. Garling [3], [4] gave an example of a nonuniformly convex Banach space with a basis which is subsymmetric but not symmetric. Professor Bill Johnson has conveyed to us that Dacunha Castelle has recently proved that in $L_p([0, 1])$, $1 < p < \infty$, every basic sequence which is subsymmetric is also symmetric. Of course, it is well known that $L_p([0, 1])$, $1 \leq p < \infty$, $p \neq 2$, has no subsymmetric basis. In this paper, a uniformly convex Banach space with a basis (δ_i) is constructed such that (δ_i) is subsymmetric but not symmetric. This substantiates a conjecture of William B. Johnson.

We follow the standard notation for Banach space theory. A basis (x_n) for a space $(X, \|\cdot\|)$ is called unconditionally monotone if $\|\sum \alpha_n x_n\| \leq \|\sum \beta_n x_n\|$ whenever $|\alpha_n| \leq |\beta_n|$. Suppose X has an unconditionally monotone basis (x_i) . The norm on X is called p -convex if

$$\left\| \sum (|\alpha_i|^p + |\beta_i|^p)^{1/p} x_i \right\|^p \leq \left\| \sum \alpha_i x_i \right\|^p + \left\| \sum \beta_i x_i \right\|^p, \quad p \geq 1,$$

for all scalars (α_i) and (β_i) . The norm on X is said to satisfy an upper l_p -estimate if $\|x + y\|^p \leq \|x\|^p + \|y\|^p$ whenever x and y have disjoint support relative to (x_i) . It is clear that if $\|\cdot\|$ is p -convex, then it satisfies the upper l_p -estimate. The norm on X is said to satisfy a lower l_q -estimate, if $\|x + y\|^q \geq \|x\|^q + \|y\|^q$ whenever x and y have disjoint support relative to (x_i) . A basis (x_i) is called subsymmetric if it is unconditional and equivalent to each of its subsequences. It is called symmetric if it is equivalent to each of its permutations.

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2. The construction of the space. Let X denote the space of all scalar valued sequences which have only finitely many nonzero coordinates. As in [4], we define a norm on X by

$$\|(\beta_n) \| = \left[\sup_{n_1 < n_2 < \dots} \sum_i \frac{\beta_{n_i}^2}{\sqrt{i}} \right]^{1/2}.$$

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It follows immediately that the norm $\|\cdot\|$ is 2-convex. From now on, let $r = 4, 8$. We define two other norms on X as follows:

$$|(\beta_n)|_r = \left(\sup_{(A_k)} \sum_k \left\| \sum_{i \in A_k} \beta_i \delta_i \right\|^r \right)^{1/r},$$

where the A_k 's are pairwise disjoint finite subsets of the integers and (δ_i) are unit vectors. It is a routine calculation to show that $(X, |\cdot|_r)$ is a Banach space.

THEOREM. *The space $(X, |\cdot|_8)$ is a uniformly convex Banach space. The unit vectors (δ_i) constitute a subsymmetric basis for X which is not symmetric.*

PROOF. We divide the proof into several steps.

Step I. It is straightforward to show that (δ_i) is an unconditionally monotone basis for X , (δ_i) is subsymmetric and that $|\cdot|_r$ satisfies the lower l_r -estimate.

Step II. We assert that the norm $|\cdot|_r$ is 2-convex. Indeed, let

$$\left| \sum_i (|\alpha_i|^2 + |\beta_i|^2)^{1/2} \delta_i \right|^2 = \left[\sum_{k=1}^j \left\| (|\alpha_i|^2 + |\beta_i|^2)^{1/2} \delta_i \right\|^r \right]^{2/r}$$

for some pairwise disjoint finite subsets $A_1, A_2, \dots, A_j$.

$$\begin{aligned} &= \left[\sum_{k=1}^j \left\| \sum_{i \in A_k} (|\alpha_i|^2 + |\beta_i|^2)^{1/2} \delta_i \right\|^r \right]^{2/r} \\ &\leq \left[\sum_{k=1}^j \left(\left\| \sum_{i \in A_k} \alpha_i \delta_i \right\|^2 + \left\| \sum_{i \in A_k} \beta_i \delta_i \right\|^2 \right)^{r/2} \right]^{2/r} \end{aligned}$$

as $\|\cdot\|$ is 2-convex,

$$\leq \left[\sum_{k=1}^j \left(\left\| \sum_{i \in A_k} \alpha_i \delta_i \right\|^2 \right)^{r/2} \right]^{2/r} + \left[\sum_{k=1}^j \left(\left\| \sum_{i \in A_k} \beta_i \delta_i \right\|^2 \right)^{r/2} \right]^{2/r}$$

as $r = 4, 8$,

$$\leq \left| \sum_i \alpha_i \delta_i \right|_r^2 + \left| \sum_i \beta_i \delta_i \right|_r^2.$$

It follows that $|\cdot|_r$ is 2-convex.

Step III. Since $(X, |\cdot|_8)$ is 2-convex and satisfies the lower l_8 -estimate, it is uniformly convex, using a theorem of Figiel and Johnson [2].

Step IV. We will now show that (δ_i) is not symmetric. For $k = 2^m$, $m = 0, 1, 2, \dots$, define

$$y_m = (k^{-1/4}, (k-1)^{-1/4}, \dots, 2^{-1/4}, 1, 0, 0, \dots)$$

and

$$\bar{y}_m = (1, 2^{-1/4}, 3^{-1/4}, \dots, (k-1)^{-1/4}, k^{-1/4}, 0, \dots).$$

It is enough [4] to show that

$$|\bar{y}_m|_8 / |y_m|_8 \rightarrow \infty \quad \text{as } m \rightarrow \infty.$$

Clearly

$$(1) \quad \begin{aligned} |\bar{y}_m|_8 &\geq (1 + 2^{-1/2} \cdot 2^{-1/2} + \dots + k^{-1/2} \cdot k^{-1/2})^{1/2} \\ &\geq (c_1 + m \cdot \log 2)^{1/2} \end{aligned}$$

for large m , c_1 being a constant.

Now we use $|\cdot|_4$ to obtain an upper estimate for $|y_m|_8$. Using the 2-convexity of $|\cdot|_4$, we get

$$(2) \quad |y_m|_4^2 \leq \sum_{j=1}^{m+1} |z_j|_4^2$$

where

$$z_j = (0, 0, \dots, (2^{m+1-j})^{-1/4}, (2^{m+1-j} - 1)^{-1/4}, \dots, (2^{m-j} + 1)^{-1/4}, 0, \dots),$$

$$1 \leq j \leq m,$$

$$z_{m+1} = (0, 0, \dots, 1, 0, \dots),$$

and

$$\text{supp}(z_j) \cap \text{supp}(z_l) = \emptyset, \quad j \neq l.$$

To find an upper estimate for $|z_j|_4$, for the sake of convenience of notation, consider

$$x_l = (0, 0, \dots, (2l)^{-1/4}, (2l-1)^{-1/4}, \dots, (l+1)^{-1/4}, 0, \dots)$$

where l is an integer. Let the norm of x_l be attained over the pairwise disjoint finite subsets $A_1, A_2, \dots, A_s$. Let

$$A_n = \{i_1^{(n)}, i_2^{(n)}, \dots, i_r^{(n)}\}, \quad i_1^{(1)} = 1, 1 \leq n \leq s.$$

Using an argument as in [4, p. 586] and the fact that $(A_n)_{n=1}^s$ gives the norm for x_l , we have $\sum_{i=1}^s r_i = l$ and

$$\begin{aligned} |x_l|_4^4 &= \sum_{n=1}^s \left[\sum_{p=1}^{r_n} \frac{1}{\sqrt{2l+1-i_p^{(n)}}} \cdot \frac{1}{\sqrt{p}} \right]^2 \\ &\leq \frac{1}{l+1} \cdot \sum_{n=1}^s \left(\sum_{p=1}^{r_n} \frac{1}{\sqrt{p}} \right)^2 \\ &\leq \frac{1}{l+1} \cdot \sum_{n=1}^s 4 \cdot r_n = \frac{4}{l+1} \cdot l \leq 4 \end{aligned}$$

i.e.

$$(3) \quad |x_l|_4^2 \leq 2.$$

Using (3) in (2), we get

$$(4) \quad |y_m|_4 \leq \sqrt{2(m+1)}.$$

It can be proved, as in [4], that

$$(5) \quad \|y_m\| \leq c_2$$

where c_2 is a constant.

Let $(B_i)_{i=1}^s$ be any pairwise disjoint finite subsets of the integers. For the sake of convenience, write $y_m = \sum \alpha_i \delta_i$. Then we have

$$\sum_{j=1}^s \left\| \sum_{i \in B_j} \alpha_i \delta_i \right\|^8 \leq \|y_m\|^4 \cdot \sum_{j=1}^s \left\| \sum_{i \in B_j} \alpha_i \delta_i \right\|^4$$

i.e.

$$\left(\sum_{j=1}^s \left\| \sum_{i \in B_j} \alpha_i \delta_i \right\|^8 \right)^{1/8} \leq (c_2)^{1/2} \cdot (|y_m|_4)^{1/2}$$

using (5),

$$\leq (c_2)^{1/2} (2(m+1))^{1/4}$$

using (4).

Since $(B_i)_{i=1}^s$ are arbitrary, we get

$$(6) \quad |y_m|_8 \leq c_3 \cdot (m+1)^{1/4}.$$

From (1) and (6), we get $|\bar{y}_m|_8 / |y_m|_8 \rightarrow \infty$ as $m \rightarrow \infty$. Q.E.D.

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