A THEOREM ON C-TOTALLY REAL MINIMAL SURFACE

SEIICHI YAMAGUCHI, MASAHIRO KON AND YASUSHI MIYAHARA

ABSTRACT. We prove the corresponding theorem for a C-totally real minimal surface to Yau's theorem for a totally real minimal surface of a Kähler surface with constant holomorphic sectional curvature.

1. Introduction. Recently, S. T. Yau [11] has dealt with a totally real minimal surface and proved

THEOREM A. *Let M be a totally real minimal surface of a Kähler surface with constant holomorphic sectional curvature C . Then:*

- (i) *If M has genus zero, M is the standard embedding of RP^2 in CP^2 .*
- (ii) *If M is a complete nonnegative curved surface, M is totally geodesic or flat.*
- (iii) *If M is complete nonpositive curved with Gauss curvature K and $C/4 - K \geq \alpha > 0$ for some constant α , then M is totally geodesic or flat.*

On the other hand [9], [10], we have introduced the notion of a C-totally real submanifold of a Sasakian manifold and proved

THEOREM B [10]. *Let $\overline{M}^{2n+1}(k)$ be a $(2n + 1)$ -dimensional Sasakian manifold with constant ϕ -holomorphic sectional curvature k and M be an n -dimensional compact C-totally real minimal submanifold of $\overline{M}^{2n+1}(k)$. If $\|A\|^2 < n(n + 1) \cdot (k + 3)/4(2n - 1)$ or, equivalently, $\rho > n^2(n - 2)(k + 3)/2(2n - 1)$, then M is totally geodesic, where ρ is the scalar curvature.*

The purpose of this paper is to obtain the corresponding theorem for a C-totally real minimal surface to Theorem A, that is, we shall show

THEOREM. *Let M be a C-totally real minimal surface of a 5-dimensional Sasakian manifold $\overline{M}(k)$ with constant ϕ -holomorphic sectional curvature k . Then:*

- (i) *If M has genus zero, M is totally geodesic.*
- (ii) *If M is a complete nonnegative curved surface, M is totally geodesic or flat.*
- (iii) *If M is complete nonpositive curved with Gauss curvature K and if $(k + 3)/4 - K \geq \alpha > 0$ for some constant α , then M is flat.*

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COROLLARY. *Let M be a C -totally real minimal surface of a 5-dimensional unit sphere S^5 . Then we have (i) and (ii) in Theorem and (iii) in Theorem can be rewritten as follows: (iii)' If M is a complete nonpositive curved surface, then M is flat.*

2. Preliminaries. Let $\bar{M}^1$ be a Riemannian manifold of dimension $n + p$ and M an n -dimensional submanifold of $\bar{M}$. Let $\langle , \rangle$ be the metric tensor field on $\bar{M}$ as well as the metric induced on M . We denote by $\bar{\nabla}$ the covariant differentiation in $\bar{M}$ and by ∇ the covariant differentiation in M determined by the induced metric on M . Let $\mathfrak{X}(\bar{M})$ (resp. $\mathfrak{X}(M)$) be the Lie algebra of vector fields on $\bar{M}$ (resp. M) and $\mathfrak{X}^\perp(M)$ the set of all vector fields normal to M .

The Gauss-Weingarten formulas are given by

$$\begin{aligned}\bar{\nabla}_X Y &= \nabla_X Y + B(X, Y), \quad \bar{\nabla}_X N = -A^N(X) + D_X N, \\ X, Y &\in \mathfrak{X}(M), N \in \mathfrak{X}^\perp(M),\end{aligned}$$

where D is the normal connection in the normal bundle. Both A and B are called the second fundamental form and they satisfy

$$\langle A^N(X), Y \rangle = \langle B(X, Y), N \rangle.$$

The Gauss equation is given by

$$\begin{aligned}\langle \bar{R}(Z, Y)X, W \rangle &= \langle R(Z, Y)X, W \rangle + \langle B(Y, X), B(Z, W) \rangle \\ &\quad - \langle B(X, Z), B(Y, W) \rangle, \quad W, X, Y, Z \in \mathfrak{X}(M).\end{aligned}$$

Now we define the covariant derivative of second fundamental form B as follows:

$$\tilde{\nabla}_X(B)(Y, Z) = D_X(B(Y, Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z)$$

for any vector fields $X, Y, Z, \in \mathfrak{X}(M)$. For the second fundamental form A , we define its covariant derivative by setting

$$\begin{aligned}\nabla_X(A)^N(Y) &= \nabla_X(A^N(Y)) - A^{D_X N}(Y) - A^N(\nabla_X Y), \\ X, Y, &\in \mathfrak{X}(M), N \in \mathfrak{X}^\perp(M).\end{aligned}$$

Clearly we see $\langle \tilde{\nabla}_X(B)(Y, Z), N \rangle = \langle \nabla_X(A)^N(Y), Z \rangle$.

The mean curvature vector H is defined by $H = n^{-1} \text{Tr } B$. A submanifold M is said to be minimal if $H = 0$ identically. Moreover M is called a totally geodesic submanifold in $\bar{M}$ if its second fundamental form B is identically zero.

3. C-totally real submanifold. Let $\bar{M}$ be a Sasakian manifold with structure tensors $(\phi, \xi, \eta, \langle , \rangle)$. Then the structure tensors satisfy the following equations:

$$\begin{aligned}\phi^2 &= -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1, \\ \bar{\nabla}_X \xi &= \phi X, \quad (\bar{\nabla}_X \phi)Y = \eta(Y)X - \langle X, Y \rangle \xi, \quad X, Y \in \mathfrak{X}(\bar{M}).\end{aligned}$$

¹ Manifolds, mappings, metrics, . . . , etc. are assumed to be differentiable and of class C^∞ .

The Sasakian manifold is odd dimensional orientable manifold. When the curvature tensor of a $(2n + 1)$ -dimensional Sasakian manifold $\overline{M}$ has the following form

$$\begin{aligned} 4\overline{R}(X, Y)Z &= (k + 3)(\langle Y, Z \rangle X - \langle X, Z \rangle Y) \\ &\quad + (k - 1)(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \langle X, Z \rangle \eta(Y)\xi \\ &\quad - \langle Y, Z \rangle \eta(X)\xi + \langle \phi Y, Z \rangle \phi X + \langle \phi Z, X \rangle \phi Y - 2\langle \phi X, Y \rangle \phi Z), \end{aligned}$$

$\overline{M}$ is called a space of constant ϕ -holomorphic sectional curvature k . In such a space, k is constant if $n > 1$. It is well known that an odd dimensional sphere is Sasakian and a Sasakian manifold is a contact manifold.

Let us recall the definition on a C -totally real submanifold in a Sasakian manifold.

Let $\overline{M}$ be a $(2n + 1)$ -dimensional contact manifold with contact form η . The Pfaffian equation $\eta = 0$ determines in $\overline{M}$ a $2n$ -dimensional distribution. It is called the contact distribution. A submanifold M in $\overline{M}$ is said to be an integral submanifold of the contact distribution if and only if any tangent vector of M belong to the contact distribution. We shall call the integral submanifold M of the contact distribution of a Sasakian manifold a C -totally real submanifold. Then we have known $\dim M \leq n$, and the following theorem has been proved [9].

THEOREM C. *Let M be an m ($m \leq n$) dimensional C -totally real submanifold in a Sasakian manifold $\overline{M}^{2n+1}$ with structure tensors $(\phi, \xi, \eta, \langle \cdot, \cdot \rangle)$. Then we have the following:*

- (i) *The second fundamental form of ξ direction is identically zero.*
- (ii) *If $X \in \mathfrak{X}(M)$, then $\phi X \in \mathfrak{X}^\perp(M)$.*
- (iii) *If $m = n$, then $A^{\phi X}(Y) = A^{\phi Y}(X)$ for $X, Y \in \mathfrak{X}(M)$.*

3. Proof of theorem. In the following we assume that $\overline{M}(k)$ is a 5-dimensional Sasakian manifold with constant ϕ -holomorphic sectional curvature and M is a C -totally real minimal surface of $\overline{M}(k)$. By the assumptions, we get the Simons' type formula for the second fundamental form A as follows:

$$(3.1) \quad \nabla^2 A = -A \cdot \tilde{A} - \underline{A} \cdot A + \frac{1}{4}(3k + 5)A^2,$$

where the operators $\tilde{A}$ and $\underline{A}$ are defined by

$$\tilde{A} = {}^t A \cdot A \quad \text{and} \quad \underline{A} = \sum_{\alpha=3}^5 \text{ad } A^\alpha \text{ ad } A^\alpha.$$

We take a frame e_1, e_2 for $T_p(M)$ and a frame $\phi e_1, \phi e_2, \xi$ for $T_p(M)^\perp$. Then the second fundamental form can be expressed as

$$(3.2) \quad A^{\phi e_1} = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \quad A^{\phi e_2} = \begin{pmatrix} 0 & -a \\ -a & 0 \end{pmatrix}, \quad A^\xi = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, owing to (3.1) and (3.2), we get

² See [9] or [10].

$$(3.3) \quad \frac{1}{2}\Delta\|A\|^2 = \langle \nabla A, \nabla A \rangle + (3k + 5)a^2 - 24a^4.$$

By the way, we can easily obtain

$$(3.4) \quad \langle \nabla A, \nabla A \rangle = 8\|da\|^2 + 4a^2$$

because of $\|A\|^2 = 4a^2$. Moreover, by the Gauss equation, it follows that

$$(3.5) \quad 2a^2 = (k + 3)/4 - K,$$

where K denotes the Gauss curvature of M , and consequently, it holds that

$$(3.6) \quad \Delta\|A\|^2 = 24a^2 K + 16\|da\|^2$$

by virtue of (3.3), (3.4) and (3.5). Using (3.6) it is easy to prove

$$(3.7) \quad \log\|A\|^2 = 6K$$

whenever $\|A\|^2 \neq 0$. Using the isothermal coordinate and (3.7), it can be proved that either $\|A\|^2 \equiv 0$ or $\|A\|^2$ vanishes only at isolated points.

Let us prove the theorem stated in §1.

If M has genus zero, then M is totally geodesic by virtue of the Gauss-Bonnet theorem and (3.7).

In the next place, we shall discuss the case of $K \geq 0$. Then M is compact or parabolic. Now assume that M is compact. Then by Theorem B, M is totally geodesic if $K > 0$. Next we suppose that M is parabolic and noncompact. Then (3.7) implies that $\log\|A\|^2$ is subharmonic. On the other hand, we have

$$\log\|A\|^2 = \log 4a^2 = \log \frac{1}{2}(k + 3 - 4K) \leq \log \frac{1}{2}(k + 3).$$

Therefore, $\log\|A\|^2$ is bounded from above, and this means that $\|A\|^2$ is constant on M . This completes the proof of the case of $K \geq 0$.

In the last place, we consider the case of $K \leq 0$. It is possible to choose a system of isothermal coordinates $\{x^1, x^2\}$ covering M such that the induced metric tensor g has the following form

$$g = E\{(dx^1)^2 + (dx^2)^2\}.$$

Let g^* denote the new metric tensor in M given by $g^* = \|A\|^{2/3}g$. Then, by (3.7), we see that the Gauss curvature K^* of M with respect to g^* vanishes. Thus, g^* is a flat metric in M which is conformally equivalent to g . Since $\|A\|^{2/3}$ is bounded from below using the assumption, the metric g^* is complete, and hence the universal covering surface $\tilde{M}$ of M is conformally equivalent to the Euclidean plane E^2 . The function $\log\|A\|^2$ is a globally defined function and is bounded from below by our assumption. Moreover, from (3.7), we see that $\log\|A\|^2$ is a superharmonic function. Lifting this to $\tilde{M}$, we obtain a superharmonic function bounded from below on $\tilde{M}$. Therefore $\tilde{M}$ is parabolic, which means that $\log\|A\|^2$ is constant. This completes the proof of the case $K \leq 0$. Summing up the results obtained above, we get the theorem, and we immediately get the corollary from the theorem.

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DEPARTMENT OF MATHEMATICS, SCIENCE UNIVERSITY OF TOKYO, TOKYO,
JAPAN