A FACTORIZATION OF THE DIRECT LIMIT OF HILBERT CUBES

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ABSTRACT. We show that the countable direct limit of Hilbert cubes Q^∞ is homeomorphic to the product of the Hilbert cube Q and the countable direct limit of lines R^∞ . As a consequence, two open subsets of R^∞ have the same homotopy type if and only if their products with Q are homeomorphic. Combined with a theorem of D. W. Henderson our result implies that $X \times Q \times R^\infty \cong Q \times R^\infty$, where X is any locally compact, separable AR (metric).

At the problem session at the Conference on Infinite-Dimensional Topology at Louisiana State University, October 1973, Tom Chapman asked if $R^\infty \times Q \cong Q^\infty$. Here $R^\infty = \text{dir lim } R^n$, $Q^\infty = \text{dir lim } Q^n$, where R denotes the reals and Q denotes the Hilbert cube, which we regard as the countably infinite product of $I = [0, 1]$. (We remark that if B is any separable, infinite-dimensional Banach space, then its conjugate B^* with the bounded weak-* topology is homeomorphic to Q^∞ [4, Theorem II-6].) We answer the above question in the affirmative. As a corollary we show (Corollary 5) that two open subsets of R^∞ have the same homotopy type iff their products with Q are homeomorphic. Finally, combining our result with work of D. W. Henderson [6] we obtain a factorization result for locally compact, separable AR's and ANR's (Theorem 7).

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Let $J = [-1, 1]$. Define $i_n: I^n \rightarrow I^{n+1}$ and $j_n: J^n \rightarrow J^{n+1}$ by $i_n(x_1, \dots, x_n) = (x_1, \dots, x_n, 0)$, $j_n(x_1, \dots, x_n) = (x_1, \dots, x_n, 0)$. Note that $i_n(I^n)$ is a Z-set in I^{n+1} , whereas $j_n(J^n)$ is not a Z-set in J^{n+1} . Let $I^\infty = \text{dir lim } \{I^n; i_n\}$, $J^\infty = \text{dir lim } \{J^n; j_n\}$.

LEMMA 1. *There is a homeomorphism $h: J^\infty \rightarrow I^\infty$ such that h/J^n is piecewise linear, $n \geq 1$, and*

$$h(J^n) = \{(x_1, \dots, x_{n+1}) \in I^{n+1} \mid x_i = 0 \text{ some } i, 2 \leq i \leq n+1\}.$$

PROOF. Let

$$J_+^n = \{(x_1, \dots, x_n) \in J^n \mid x_n \geq 0\}, \quad J_-^n = \{(x_1, \dots, x_n) \in J^n \mid x_n \leq 0\}.$$

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Note [8, 3.15, p. 36] that any p.l. (piecewise linear) homeomorphism $J^n \rightarrow \partial I^{n+1}$ extends to a p.l. homeomorphism $J_+^{n+1} \rightarrow I^{n+1}$.

Inductively, we define p.l. embeddings $h_n: J^n \rightarrow I^{n+1}$ such that

$$h_n(J^n) = \{(x_1, \dots, x_{n+1}) \in I^{n+1} \mid x_i = 0, \text{ some } i, 2 \leq i \leq n+1\},$$

and such that h_{n+1} is an extension of h_n . Define $h_1: J^1 \rightarrow I^2$ by $h_1(x_1) = ((x_1 + 1)/2, 0)$. Having defined h_n , let $h'_n: J_+^{n+1} \rightarrow I^{n+1}$ be a p.l. homeomorphism extending h_n . Define h_{n+1} by

$$h_{n+1}(x_1, \dots, x_{n+1}) \begin{cases} (h'_n(x_1, \dots, x_{n+1}), 0) & \text{on } J_+^{n+1}, \\ (h_n(x_1, \dots, x_n), -x_{n+1}) & \text{on } J_-^{n+1}. \end{cases}$$

The desired homeomorphism $h: J^\infty \rightarrow I^\infty$ can now be defined by $h/J^n = h_n$.

Let $B_n = [-n, n]^n$, $C_n = [0, n]^n$. Define $\phi_n^k: B_n^k \rightarrow B_n^{k+1}$, $\psi_n^k: C_n^k \rightarrow C_n^{k+1}$ by $\phi_n^k(x_1, \dots, x_k) = (x_1, \dots, x_k, 0)$, $x_i \in B_n$, $\psi_n^k(x_1, \dots, x_k) = (x_1, \dots, x_k, 0)$, $x_i \in C_n$. Let $B_n^\infty = \text{dir lim}_k \{B_n^k; \phi_n^k\}$, $C_n^\infty = \text{dir lim}_k \{C_n^k; \psi_n^k\}$.

LEMMA 2. For each $n \geq 1$ there is a homeomorphism $g_n: C_n^\infty \times Q \rightarrow Q^\infty$ such that $g_n(C_n^k \times Q) = Q^k$, all $k \geq 1$.

PROOF. Fix n . Let $\delta_k = (\psi_n^k \times \text{id}): C_n^k \times Q \rightarrow C_n^{k+1} \times Q$, $k \geq 1$. Let $\epsilon_k: C_n^k \times Q \rightarrow Q^k$, $k \geq 1$, be any sequence of homeomorphisms such that $\epsilon_{k+1}\delta_k(C_n^k \times Q)$ is a Z-set in Q^{k+1} . Consider

$$\begin{array}{ccc} C_n^k \times Q & \xrightarrow{\delta_k} & C_n^{k+1} \times Q \\ \downarrow \epsilon_k & & \downarrow \epsilon_{k+1} \\ Q^k & \xrightarrow{j_k} & Q^{k+1} \end{array}$$

where each $j_k: Q^k \rightarrow Q^{k+1}$ is the inclusion. Both $A_2 = \epsilon_2\delta_1(C_n^1 \times Q)$ and $B_2 = j_1\epsilon_1(C_n^1 \times Q)$ are Z-sets in Q^2 . By a theorem of Anderson [1, Corollary 10.3] we can extend $j_1\epsilon_1\delta_1^{-1}\epsilon_2^{-1}: A_2 \rightarrow B_2$ to a homeomorphism $G_2: Q^2 \rightarrow Q^2$. By replacing ϵ_2 with $G_2\epsilon_2$, we may thus assume $j_1\epsilon_1 = \epsilon_2\delta_1$. Doing this for each ϵ_k , $k \geq 2$, we obtain a homeomorphism $\epsilon: \text{dir lim}_k \{C_n^k \times Q; \delta_k\} \rightarrow Q^\infty$. Arguing as in the proof of Theorem 4.4, p. 263 of [2], we see that $C_n^\infty \times Q = \text{dir lim}_k \{C_n^k \times Q; \delta_k\}$, and we can take $g_n = \epsilon$.

LEMMA 3. Let X, Y be subsets of Q^∞ such that there exist arbitrarily large integers r and s such that $X \cap Q^r$ is a Z-set in Q^r and $Y \cap Q^s$ is a Z-set in Q^s . Then any homeomorphism $f: X \rightarrow Y$ extends to a homeomorphism $f': Q^\infty \rightarrow Q^\infty$.

PROOF. For $n \geq 1$ and $A \subset Q^\infty$, let $A_n = A \cap Q^n$.

Inductively, suppose we have integers $r_1, \dots, r_n, s_1, \dots, s_n$ and homeomorphisms $f_1, \dots, f_n$ such that $f_i: Q^{r_i} \rightarrow Q^{s_i}$, $i = 1, \dots, n$, and (a) $f_i/X_{r_i} = f/X_{r_i}$, $i = 1, \dots, n$, (b) $f_i^{-1}(Y_{s_{i-1}}) \subset X_{r_i}$, $i = 2, \dots, n$, (c) $F_i(Q^{r_{i-1}}) \cap Y_{s_i} = f(X_{r_{i-1}})$, $i = 2, \dots, n$, and

$$(d) f_i/[Q^{r_{i-2}} \cup f_{i-1}^{-1}(Q^{s_{i-2}})] = f_{i-1}/[Q^{r_{i-2}} \cup f_{i-1}^{-1}(Q^{s_{i-2}})], \quad i = 3, \dots, n.$$

(We assume $n \geq 3$. The constructions of f_1, f_2, f_3 should be clear from the general inductive step.) Choose $r_{n+1} > r_n$ such that $f^{-1}(Y_{s_n}) \subset X_{r_{n+1}}$ and $X_{r_{n+1}}$ is a Z -set in $Q^{r_{n+1}}$. Then choose $s_{n+1} > s_n$ such that $f(X_{r_{n+1}}) \subset Y_{s_{n+1}}$ and $Y_{s_{n+1}}$ is a Z -set in $Q^{s_{n+1}}$. Let $B = Q^{s_n} \cup Y_{s_{n+1}}$, $C = Q^{s_{n-1}} \cup f_n(Q^{r_{n-1}}) \cup f_n(X_{r_n})$. Note that C is a closed subset of B , and B is a Z -set in $Q^{s_{n+1}}$. Let

$$D = \{x = \{x_i\} \in Q \mid x_1 = x_2 = 1\},$$

$$E = \{x \in Q \mid x_1 = 1\} \cup \{x \in Q \mid x_2 = 1\}.$$

Let $h: Q^{s_{n+1}} \rightarrow Q$ be a homeomorphism such that $h(B) \subset D$ [1, Theorem 8.1 and Corollary 10.3]. Let $\phi: D \rightarrow [0, \frac{1}{2}]$ be a continuous map such that $\phi^{-1}(0) = h(C)$. Given $d \in D$, define $\delta_d: [0, 2] \rightarrow [0, 2]$ by

$$\delta_d(t) = \begin{cases} t(1 + \phi(d)), & t \in [0, 1], \\ 1 + \phi(d) + (t - 1)(1 - \phi(d)), & t \in [1, 2]. \end{cases}$$

Define $\rho: E \rightarrow D$ by $\rho(x) = (1, 1, x_3, x_4, \dots)$. Define $g: E \rightarrow E$ by

$$g(x) = \begin{cases} (\min\{1, \delta_{\rho(x)}(x_1)\}, 1 - \max\{0, \delta_{\rho(x)}(x_1) - 1\}, x_3, x_4, \dots) & \text{if } x_2 = 1, \\ (1, 2 - \delta_{\rho(x)}(2 - x_2), x_3, x_4, \dots) & \text{if } x_1 = 1. \end{cases}$$

Then g is a homeomorphism “sliding points around the corner D ”, and for $x \in D$, $g(x) \in D \Leftrightarrow x \in h(C) \Leftrightarrow g(x) = x$. Since E is a Z -set in Q , we can extend g to a homeomorphism $g': Q \rightarrow Q$. Let $f'_n = h^{-1}g'hf_n: Q^{r_n} \rightarrow Q^{s_{n+1}}$. Then f'_n is an embedding such that (i) f'_n agrees with f_n on $f_n^{-1}(Q^{s_{n-1}}) \cup Q^{r_{n-1}} \cup X_{r_n}$, (ii) $f'_n(Q^{r_n})$ is a Z -set in $Q^{s_{n+1}}$, and (iii) $f'_n(Q^{r_n}) \cap Y_{s_{n+1}} = f(X_{r_n})$. To prove (iii), let

$$y = f'_n(x) = (h^{-1}g')(hf_n)(x) \in f'_n(Q^{r_n}) \cap Y_{s_{n+1}}.$$

Then $hf_n(x) \in D$ and $g'hf_n(x) = ghf_n(x) \in h(Y_{s_{n+1}})$ imply that $ghf_n(x) = hf_n(x)$ so that $h(f_n(x)) \in h(C)$. Thus,

$$\begin{aligned} f'_n(x) &= f_n(x) \in Y_{s_{n+1}} \cap C \\ &= Y_{s_{n-1}} \cup [Y_{s_n} \cap f_n(Q^{r_{n-1}})] \cup [Y_{s_{n+1}} \cap f_n(X_{r_n})] \subset f_n(X_{r_n}) \end{aligned}$$

by induction hypotheses (b) and (c). The other containment is clear. Because of (i) and (iii) above we may extend f'_n to an embedding $f''_n: Q^{r_n} \cup X_{r_{n+1}} \rightarrow Q^{s_{n+1}}$ by defining $f''_n/X_{r_{n+1}} = f/X_{r_{n+1}}$. Since the image of f''_n is a Z -set in $Q^{s_{n+1}}$, we may extend [1, Corollary 10.3] f''_n to a homeomorphism $f_{n+1}: Q^{r_{n+1}} \rightarrow Q^{s_{n+1}}$ satisfying our induction hypotheses.

By induction we obtain f_n , all n . The desired homeomorphism $f': Q^\infty \rightarrow Q^\infty$ can now be defined by $f'/Q^{r_n} = f_{n+1}/Q^{r_n}$.

THEOREM 4. $R^\infty \times Q \cong Q^\infty$.

PROOF. Fix $n \geq 1$. Define $\beta_n: B_n^\infty \rightarrow J^\infty$, $\gamma_n: C_n^\infty \rightarrow I^\infty$ by

$$\beta_n(x_1, x_2, \dots) = (1/n)(x_1, \dots, x_{1_n}, x_2, \dots, x_{2_n}, \dots)$$

(here $x_i = (x_{i_1}, \dots, x_{i_n}) \in B_n$),

$$\gamma_n(x_1, x_2, \dots) = (1/n)(x_{1_1}, \dots, x_{1_n}, x_{2_1}, \dots, x_{2_n}, \dots).$$

Define $\alpha_n: B_n^\infty \rightarrow B_{n+1}^\infty$ by

$$(x_{1_1}, \dots, x_{1_n}, x_{2_1}, \dots, x_{2_n}, \dots) \rightarrow (x_{1_1}, \dots, x_{1_n}, 0, x_{2_1}, \dots, x_{2_n}, 0, \dots).$$

Let $h_n = (\gamma_n^{-1} h \beta_n \times \text{id}): B_n^\infty \times Q \rightarrow C_n^\infty \times Q$, where h is the homeomorphism of Lemma 1. Clearly h_n is a homeomorphism. Consider

$$\begin{array}{ccc} B_n^\infty \times Q & \xrightarrow{\alpha_n \times \text{id}} & B_{n+1}^\infty \times Q \\ \downarrow g_n h_n & & \downarrow g_{n+1} h_{n+1} \\ Q^\infty & \xrightarrow{\rho} & Q^\infty \end{array}$$

Here g_n is the homeomorphism of Lemma 2, and ρ is defined by $\rho(x_1, x_2, \dots) = (x_1, 0, x_2, 0, \dots)$, $x_i \in Q$. Let

$$X = g_{n+1} h_{n+1} (\alpha_n \times \text{id})(B_n^\infty \times Q), \quad Y = \rho g_n h_n (B_n^\infty \times Q).$$

Clearly $Y \cap Q^k$ is a Z -set in Q^k , all large k . We now show this is also true for X :

$$\begin{aligned} X \cap Q^k &= g_{n+1} [(\gamma_{n+1}^{-1} h \beta_{n+1} \alpha_n (B_n^\infty) \cap C_{n+1}^k] \times Q \\ &= g_{n+1} [(\gamma_{n+1}^{-1} h (J^n \times 0 \times J^n \times 0 \times \dots) \cap \gamma_{n+1}^{-1} (I^{kn+k})) \times Q] \end{aligned}$$

which is a closed subset of $g_{n+1} [\gamma_{n+1}^{-1} (h(J^{kn+k-1})) \times Q]$ since $h(J^r) \supset I^r$. Since $h(J^r)$ is a Z -set in I^{r+1} , it follows that $X \cap Q^k$ is a Z -set in $g_{n+1} [\gamma_{n+1}^{-1} (I^{kn+k}) \times Q] = Q^k$. Thus, applying our Lemma 3 as Corollary 10.3 of [1] was applied in the proof of Lemma 2, we obtain a homeomorphism

$$\text{dir} \lim_n \{B_n^\infty \times Q; \alpha_n \times \text{id}\} \rightarrow \text{dir} \lim \{Q^\infty; \rho\}.$$

We finish the proof by showing that the first direct limit (above) is $R^\infty \times Q$, the latter Q^∞ . A proof analogous to the proof of Theorem 4.4, p. 263 of [2] shows that

$$\text{dir} \lim_n \{B_n^\infty \times Q; \alpha_n \times \text{id}\} \cong \left(\text{dir} \lim_n \{B_n^\infty; \alpha_n\} \right) \times Q.$$

Define $\mu_n: B_n^\infty \rightarrow (R^\infty)^\infty$ by

$$\begin{aligned} \mu_n((x_{1_1}, \dots, x_{1_n}, x_{2_1}, \dots, x_{2_n}), \dots) \\ = ((x_{1_1}, \dots, x_{1_n}, 0, 0, \dots), (x_{2_1}, \dots, x_{2_n}, 0, 0, \dots), \dots). \end{aligned}$$

Define $\mu: \text{dir} \lim_n \{B_n^\infty; \alpha_n\} \rightarrow (R^\infty)^\infty$ by $\mu/B_n^\infty = \mu_n$. By Proposition II-1(a) and Corollary III-3 of [4], sets of the form

$$[(\mathcal{C}_{1_1} \times \mathcal{C}_{1_2} \times \dots) \times (\mathcal{C}_{2_1} \times \mathcal{C}_{2_2} \times \dots) \times \dots] \cap (R^\infty)^\infty,$$

$\mathcal{O}_{ij}$ open in R , are a basis for $(R^\infty)^\infty$. It follows easily that μ is continuous. By Corollary III-2 of [4], μ^{-1} is continuous if it is continuous on each $(R^k)^n$ (and hence if continuous on each $(B_k)^n$), and this is clear. Finally [4, Corollary III-3], $(R^\infty)^\infty \cong R^\infty$ so that, indeed,

$$\operatorname{dir} \lim_n \{B_n^\infty \times Q; \alpha_n \times \operatorname{id}\} \cong R^\infty \times Q.$$

Define $\tau_n: (Q^\infty)^n \rightarrow Q^\infty$ by

$$\tau_n(\{x_{1j}, \dots, \{x_{nj}\}) = (x_{11}, x_{21}, \dots, x_{n1}, x_{12}, \dots, x_{n2}, \dots).$$

Consider

$$\begin{array}{ccc} (Q^\infty)^n & \xrightarrow{\lambda} & (Q^\infty)^{n+1} \\ \downarrow \tau_n & & \downarrow \tau_{n+1} \\ Q^\infty & \xrightarrow{\rho} & Q^\infty \end{array}$$

where λ is inclusion. Then $\tau_{n+1} \lambda[(Q^\infty)^n]$ and $\rho \tau_n[(Q^\infty)^n]$ satisfy the hypotheses of Lemma 3. As before, by applying our Lemma 3, we obtain a homeomorphism of direct limits, $(Q^\infty)^\infty \cong \operatorname{dir} \lim \{Q^\infty; \rho\}$. Since $(Q^\infty)^\infty \cong Q^\infty$ [4, Corollary III-3], this completes the proof of Theorem 4.

COROLLARY 5. *Let U, V be open subsets of R^∞ . Then U and V have the same homotopy type iff $U \times Q$ is homeomorphic to $V \times Q$.*

PROOF. By Theorem 4, $U \times Q$ and $V \times Q$ are open subsets of Q^∞ , and as such they are homeomorphic iff they have the same homotopy type [5, Theorem II-1 and Corollary II-12].

LEMMA 6. *Let X be a locally compact, separable ANR (metric). Then X embeds as a closed subset of Q^∞ .*

PROOF. Since X is Lindelöf and locally compact, we can write $X = \bigcup \{X_i | i = 1, 2, \dots\}$ where each X_i is a compact, metric subspace and $X_i \subset \operatorname{int} X_{i+1}$. Let $h_1: X_1 \rightarrow Q^1$ be any (closed) embedding [2, p. 195]. Suppose, inductively, that we have embeddings $h_i: X_i \rightarrow Q^i, i = 1, \dots, n$, such that $h_i/X_{i-1} = h_{i-1}, i > 2$, and $h_n(X_n) \cap Q^i = h_n(X_i), i < n$. Let $g_{n+1}: X_{n+1} \rightarrow Q^n \subset Q^{n+1}$ be any embedding. By [1, Corollary 10.3] the induced homeomorphism $g_{n+1}(X_n) \rightarrow h_n(X_n)$ extends to a homeomorphism $g: Q^{n+1} \rightarrow Q^{n+1}$. Then $f_{n+1} = gg_{n+1}: X_{n+1} \rightarrow Q^{n+1}$ is an embedding such that $f_{n+1}/X_n = h_n$. Proceeding as in the proof of Lemma 3 with $B = Q^n \cup f_{n+1}(X_{n+1})$ and $C = f_{n+1}(X_n)$, we can now adjust f_{n+1} to obtain an embedding $h_{n+1}: X_{n+1} \rightarrow Q^{n+1}$ such that $h_{n+1}/X_n = h_n$ and $h_{n+1}(X_{n+1}) \cap Q^n = h_{n+1}(X_n)$. In this way we obtain h_n , all n . The desired closed embedding $h: X \rightarrow Q^\infty$ can now be defined by $h/X_n = h_n$.

THEOREM 7. (a) *If X is a locally compact, separable AR (metric), then $X \times Q \times R^\infty \cong Q \times R^\infty$.* (b) *If X is a locally compact, separable ANR (metric), then $X \times Q \times R^\infty$ embeds as an open subset of $Q \times R^\infty$.*

PROOF. (If it were known that Q^∞ is an AR for perfectly normal spaces, a proof analogous to the proof of Theorem 2 of [6] would work here.) By [4,

Theorem II-6] we may identify $Q \times R^\infty \cong Q^\infty$ with L , by which we denote a separable, infinite-dimensional Hilbert space with its bounded weak topology. Since L is a topological vector space [3, Corollary 5, p. 428] and $L^\infty \cong L$ [4, Corollary III-3], the proof of Henderson [6] shows that if X is a retract of L , then $X \times L \cong L$.

Now suppose X is a locally compact, separable ANR (metric). By Lemma 6 we may regard X as a closed subspace of L . By [7, Theorem 4.1, p. 86] X is also an ANR (perfectly normal). By Proposition III-1 of [5] L is normal, and the proof of that same proposition shows that each open subset of L is an F_σ so that L is perfectly normal. Thus, there is a neighborhood U of X in L and a retraction $r: U \rightarrow X$. Choose open sets V and W in L such that $X \subset V \subset \bar{V} \subset W \subset \bar{W} \subset U$, and then a continuous map $\phi: L \rightarrow [0, 1]$ such that $\phi^{-1}(0) = X$ and $\phi/(L \setminus V) = 1$. Let

$$Y_1 = \{(x, t) \in \bar{W} \times R \mid t \geq \phi(x)\}, \quad Y_2 = \{(x, t) \in L \times R \mid t \leq \phi(x)\},$$

and $Y_3 = Y_1 \cap Y_2$. Define $r_1: Y_1 \rightarrow X \times [0, \infty)$ and $r_2: Y_3 \cup (L \times \{0\}) \rightarrow L \times \{0\}$ by $r_1(x, t) = (r(x), t - \phi(x))$, $r_2/(L \times \{0\}) = \text{id}$, and $r_2/Y_3 = r_1$. (This is where $\phi^{-1}(0) = X$ is used.) It follows easily from the Banach-Dieudonné theorem [3, p. 427] that $L \times R \cong L$. Thus, we may extend r_2 to a continuous map $r'_2: L \times R \rightarrow L \times \{0\} \subset L \times R$ [5, Proposition III-3]. Let $X' = (X \times [0, \infty)) \cup (L \times \{0\})$. Define $\lambda: Y_1 \cup Y_2 \rightarrow X'$ by $\lambda/Y_1 = r_1$ and $\lambda/Y_2 = r'_2$. Then λ is a retraction. Since $Y_1 \cup Y_2 \supset (W \times R) \cup [(L \setminus \bar{V}) \times (-\infty, 1)]$, a neighborhood of X' , we thus have that X' is a neighborhood retract of $L \times R$. Also, X' is contractible, and the same proof that shows that a contractible ANR is an AR now shows that X' is a retract of $L \times R \cong L$. Thus, by Henderson's theorem, $X' \times L \cong L$ so that $X \times L \cong X \times (0, \infty) \times L$ is open in L . This establishes (b). Since contractible open subsets of Q^∞ are homeomorphic to Q^∞ [5, Corollary II-12], (a) now follows immediately.

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