p -HELSON SETS

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ABSTRACT. Every infinite compact abelian group G contains a subset which is a set of interpolation for each space $m_p(G)$ ($1 < p < 2$), but which is not Helson.

In [10], Stafney proved two theorems relating $\Lambda(q)$ sets of integers to subsets of the circle group which are sets of interpolation for a certain space of multipliers. Stafney called these latter sets W_p -continuity sets and, using one of his theorems, was able to show the existence of $W_{4/3}$ -continuity sets which are not W_p -continuity sets for any $p < 4/3$. In this note we point out another relationship between $\Lambda(q)$ sets and the sets of Stafney, which we prefer to call p -Helson sets (Lemma 2). Our result is valid in any compact abelian group G , and it enables us to prove that if G is infinite, there always exist non-Helson sets which are p -Helson for every $p \in (1, 2)$. (For the definition of Helson set, and for definitions of other terms we shall use, the reader may consult [8].)

Let G be a compact abelian group, and let $\hat{G}$ be the character group of G . Let $\mathfrak{T}$ denote the set of trigonometric polynomials on G . For $p \in [1, 2]$, we define the m_p norm on the linear space $\mathfrak{T}$ by setting $\|t\|_{m_p}$ equal to the norm of $\hat{t}$ (Fourier transform of $t \in \mathfrak{T}$) as a convolution operator on $l^p(\hat{G})$. Then it is well known that $\|t\|_{m_1}$ is the norm of t in $A(G)$, the algebra of functions on G having absolutely convergent Fourier series, and that $\|t\|_{m_2}$ is just $\|t\|_\infty$, the supremum norm of t over G . It follows from the Riesz-Thorin theorem that

$$(1) \quad \|t\|_{m_{p_1}} \geq \|t\|_{m_{p_2}} \quad \text{if } 1 \leq p_1 \leq p_2 \leq 2,$$

and so

$$\|t\|_{m_p} \geq \|t\|_\infty \quad \text{for } p \in [1, 2].$$

Thus the completion of $\mathfrak{T}$ with respect to any m_p norm is a space of continuous functions on G , which we denote $m_p(G)$. Of course $m_1(G)$ is $A(G)$, while $m_2(G)$ is $C(G)$, the set of continuous functions on G .

DEFINITION 1. Let E be a closed subset of G and fix $p \in [1, 2)$. Then E is a p -Helson set if $m_p(G)|_E = C(E)$. That is, E is p -Helson if every continuous function on E extends to a function in $m_p(G)$. We mention that our definition of p -Helson set is stronger than that of Gregory [6].

Since (1) implies that $m_{p_1}(G) \subseteq m_{p_2}(G)$ if $1 \leq p_1 \leq p_2 < 2$, it is clear that if E is p_1 -Helson, then E is also p_2 -Helson. Thus if E is a true Helson set (i.e., if E is 1-Helson), then E is p -Helson for every $p \in [1, 2)$. Theorem 3 below

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shows that the converse does not hold. The following lemma will be used in the proof of Theorem 3. As in [8], we let G_d denote the group G equipped with the discrete topology.

LEMMA 2. *Let E be a closed subset of G and suppose that $p \in (1, 2)$, $1/p + 1/p' = 1$.*

- (a) *If E is Helson, then E is a Sidon subset of G_d .*
- (b) *If E is a countable Sidon subset of G_d , then E is Helson.*
- (c) *If E is p -Helson, then E is a $\Lambda(p')$ subset of G_d .*
- (d) *If E is a countable $\Lambda(p')$ subset of G_d , then E is p -Helson.*

PROOF. We prove only (c) and (d), for (a) and (b) are well known. We start by recalling a fact about the space A_p . Let $b\hat{G}$ denote the Bohr compactification of $\hat{G}$, so that $b\hat{G}$ is the character group of G_d . Let $A_p(b\hat{G})$ be the Banach space of continuous functions on $b\hat{G}$ which is defined in [3]. Then it follows from [4, Theorem 6] that E is a $\Lambda(p')$ subset of G_d if and only if there exists $\delta > 0$ such that

$$(2) \quad \|s\|_{A_p(b\hat{G})} \geq \delta \|s\|_{A(b\hat{G})}$$

for all E -polynomials s on $b\hat{G}$. On the other hand, an application of elementary duality theory shows that E is a p -Helson subset of G if and only if there exists $\delta > 0$ such that

$$(3) \quad \sup_{\|\lambda\|_{m_p(G)}=1} \left| \int_G r \, d\lambda \right| \geq \delta \|\lambda\|$$

for every measure λ concentrated on E .

Now let $x_1, \dots, x_n \in E$ be fixed and let $c_1, \dots, c_n$ be complex numbers. Let s be the E -polynomial (on $b\hat{G}$) given by $s = \sum_{i=1}^n c_i x_i$, and let λ be the measure $\sum_{i=1}^n c_i \delta_{x_i}$ on E . Then

$$(4) \quad \|s\|_{A(b\hat{G})} = \sum_{i=1}^n |c_i| = \|\lambda\|.$$

Also, we claim that

$$(5) \quad \|s\|_{A_p(b\hat{G})} = \sup_{\|\lambda\|_{m_p(G)}=1} \left| \int_G r \, d\lambda \right|.$$

Since (c) and (d) follow easily from (2)–(5), to complete the proof of the lemma it suffices to establish (5). But the dual space of $m_p(G)$ is $B_p(\hat{G})$, a space of functions on $\hat{G}$ described in [5] and [7], and it is easy to see that the RHS of (5) is $\|\hat{\lambda}\|_{B_p(\hat{G})}$. Since $\|\hat{\lambda}\|_{B_p(\hat{G})} = \|s\|_{A_p(b\hat{G})}$ by [9, Théorème 3], (5) follows.

THEOREM 3. *Let G be an infinite compact abelian group. Then there exists a closed subset E of G which is p -Helson for every $p \in (1, 2)$, but which is not Helson.*

PROOF. Let H be any infinite closed metric subgroup of G . By our Lemma 2 and by Corollary 2.2 of [2], it suffices to produce a closed countable subset E of H which, considered as a subset of H_d , is $\Lambda(s)$ for every s , but not Sidon.

Let $P = \{x_n\}_{n=1}^\infty$ be any sequence in H which converges to 0 and which

satisfies for every m the independence condition

$$\sum_{i=1}^m \varepsilon_i x_i = 0, \quad \varepsilon_i \in \{0, \pm 1\}, \quad \text{only if } \varepsilon_i = 0 \quad (1 \leq i \leq m).$$

Such a sequence is easily constructed by induction. Then P and $P + P$ are $\Lambda(s)$ sets in H_d for every s [1, Chapitre II, Théorème 5], but $P + P$ is not a Sidon set [2, Theorem 3.5]. Since the union of a finite number of $\Lambda(s)$ sets is again $\Lambda(s)$, it follows that $E = \{0\} \cup P \cup (P + P)$ is a closed countable subset of H which, as a subset of H_d , is $\Lambda(s)$ for every s but not Sidon.

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