ON BLASCHKE PRODUCTS DIVERGING EVERYWHERE ON THE BOUNDARY OF THE UNIT DISC

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ABSTRACT. If the moduli of the zeros of a Blaschke product increase sufficiently slowly the arguments of the zeros may be so chosen that the product diverges everywhere on $\{z: |z| = 1\}$.

A sequence of complex numbers $\{a_n\}$ in the unit disc is a Blaschke sequence if and only if $\sum_n (1 - |a_n|) < \infty$. If the first m terms of $\{a_n\}$ are zero and the remaining terms are nonzero, then the corresponding Blaschke product is defined by the formula

$$B(z, \{a_n\}) = z^m \prod_{n>m} b(z, a_n) = z^m \prod_{n>m} \frac{\bar{a}_n(a_n - z)}{|a_n|(1 - z\bar{a}_n)}.$$

The Blaschke product converges absolutely for $|z| < 1$, and, if

$$(1) \quad \sum_n (1 - |a_n|) \log \frac{1}{1 - |a_n|}$$

converges, Frostman [1] has shown that it also converges absolutely almost everywhere on $\{z: |z| = 1\}$. It will be shown here that if (1) does not converge then the arguments of the terms of the sequence $\{a_n\}$ can be so chosen that $B(z, \{a_n\})$ diverges whenever $|z| = 1$. Although I am not aware of the existence, in the published literature, of examples of Blaschke products which diverge everywhere on $\{z: |z| = 1\}$, some unpublished examples have been obtained by Clunie and Piranian.

THEOREM. Let $\{r_n\}$ be a positive Blaschke sequence such that

$$\sum_{n=1}^{\infty} (1 - r_n) \log(1/(1 - r_n))$$

diverges, and let

$$\theta_N = \sum_{n=1}^N (1 - r_n) \log \frac{1}{1 - r_n} \quad (N = 1, 2, 3, \dots).$$

Then if $a_n = r_n e^{i\theta_n}$ the Blaschke product $B(z, \{a_n\})$ diverges for $z = e^{i\theta}$ ($0 \leq \theta < 2\pi$).

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For each natural number k we choose $N (= N(k))$ to satisfy the inequality $\theta_{N-1} \leq \theta + 2k\pi < \theta_N$, and, since $\theta_n - \theta_{n-1} \leq 1/e$, we can then choose $M (= M(N))$ so that

$$(2) \quad \frac{1}{2} < \theta_N - \theta_M < 1.$$

Clearly $\{N(k)\}$ and $\{M(N(k))\}$ are unbounded increasing sequences.

Elementary properties of the Blaschke factors show that

$$b(e^{i\theta}, a_n) = \left(\frac{r_n e^{\frac{1}{2}i(\pi - \theta + \theta_n)} + e^{-\frac{1}{2}i(\pi - \theta + \theta_n)}}{|1 - \bar{a}_n e^{i\theta}|} \right)^2 = e^{-2i\varphi_n},$$

where

$$\varphi_n = \arctan \frac{1 - r_n}{(1 + r_n) \tan \frac{1}{2}(\theta - \theta_n)}.$$

We need to show that $\sum_{n=1}^{\infty} \varphi_n$ diverges.

We consider φ_n on the set of integers n that satisfy $M \leq n < N$. For such n we have

$$\tan \frac{1}{2}(\theta - \theta_n) < \tan \frac{1}{2}(\theta_N - \theta_n) < \theta_N - \theta_n$$

since $0 < \theta_N - \theta_n < 1$, whence

$$\varphi_n > \arctan \frac{1 - r_n}{2(\theta_N - \theta_n)}.$$

If

$$(3) \quad (1 - r_n)/(\theta_N - \theta_n) > 2,$$

then $\varphi_n > \pi/4$. On the other hand, if (3) does not hold then

$$\begin{aligned} \varphi_n &> \frac{1 - r_n}{4(\theta_N - \theta_n)} = \frac{\theta_n - \theta_{n-1}}{4 \log(1/(1 - r_n))(\theta_N - \theta_n)} \\ &> \frac{1}{4 \log(1/(1 - r_n))} \log \left(1 + \frac{\theta_n - \theta_{n-1}}{\theta_N - \theta_n} \right) \\ &= \frac{\log(1/(\theta_N - \theta_n)) - \log(1/(\theta_N - \theta_{n-1}))}{4 \log(1/(1 - r_n))} \\ &\geq \frac{\log(1/(\theta_N - \theta_n))}{4 \log(1/(1 - r_{n+1}))} - \frac{\log(1/(\theta_N - \theta_{n-1}))}{4 \log(1/(1 - r_n))}. \end{aligned}$$

Thus we have $\sum_{n=M}^{N-1} \varphi_n > \pi/4$ if (3) holds for any integer n in $[M, N - 1]$. Otherwise we have

$$\begin{aligned} \sum_{n=M}^{N-1} \varphi_n &> \frac{\log(1/(\theta_N - \theta_{N-1}))}{4 \log(1/(1 - r_N))} - \frac{\log(1/(\theta_N - \theta_{M-1}))}{4 \log(1/(1 - r_M))} \\ &> \frac{1}{4} - \frac{\log \log(1/(1 - r_N))}{4 \log(1/(1 - r_N))} - \frac{\log 2}{4 \log(1/(1 - r_M))} \end{aligned}$$

since $\theta_N - \theta_{M-1} > \frac{1}{2}$ by (2). Since $\lim_{n \rightarrow \infty} r_n = 1$, it follows that $\sum_{n=1}^{\infty} \varphi_n$ diverges.

REFERENCE

1. O. Frostman, *Sur les produits de Blaschke*, Kungl. Fysiografiska Sällskapet i Lund Förhandlingar **12** (1942), no. 15, 169–182. MR **6**, 262.

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