SEPARATION FOR DIFFERENTIAL OPERATORS AND THE L^p SPACES

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ABSTRACT. The differential operator

$$My = y^{(n)} + q_{n-1}y^{(n-1)} + \cdots + q_1y' + q_0y$$

is said to be separated in $L^p(0, \infty)$ if $y \in L^p(0, \infty)$, $y^{(n-1)}$ absolutely continuous, and $My \in L^p(0, \infty)$ imply that $q_i y^{(i)} \in L^p(0, \infty)$ for all $i = 0, 1, \dots, n-1$. As a special case of our main result we obtain: M is separated in $L^p(0, \infty)$ if $q_i = b_i + s_i$ where b_i is essentially bounded and $s_i \in L^{p_i}(0, \infty)$ for some $p_i \geq p$, $i = 0, 1, \dots, n-1$. The case $n = 2$, $p = 2$, $q_1 \equiv 0$ is due to Everitt-Giertz-Weidmann [2]. In the same paper these authors show that this result is best possible in the sense that for any p satisfying $1 < p < 2$ a function q exists in $L^p(0, \infty)$ such that $y'' + qy$ is not separated in $L^2(0, \infty)$.

1. Introduction. In this paper we investigate the question of separation for ordinary linear differential operators of the type

$$My = y^{(n)} + q_{n-1}y^{(n-1)} + \cdots + q_1y' + q_0y.$$

Throughout this paper we will be working on the interval $[0, \infty)$. So all the functions considered will be assumed to be at least locally L^1 and all our L^p spaces are on $[0, \infty)$.

We say that the operator M is separated in L^p if $y \in L^p$, $y^{(n-1)}$ absolutely continuous, and $My \in L^p$ imply that $q_i y^{(i)} \in L^p$ for $i = 0, 1, \dots, n-1$.

In [2] Everitt, Giertz and Weidmann have shown that $My = y'' + qy$ is separated in L^2 if $q \in L^p$ with $p \geq 2$ and that this result is best possible in the sense that if $1 \leq p < 2$ there exists a coefficient q_p in L^p such that $y'' + q_p y$ is not separated in L^2 . As a consequence of our main result we will see that $y^{(n)} + qy$ is separated in L^p for any $n \geq 1$ if $q \in L^{p_1}$ for some $p_1 \geq p$. The proof given in [2] relies heavily on second order techniques and so does not seem to generalize to higher orders. We base our proof on some well-known a priori estimates for the bounded coefficient case and a decomposition of an L^p function q into $q = b + r$ where b is bounded and r is "small."

A motivation for the consideration of separation is discussed in [3, §1]. The question of separation without the assumption $q_i \in L^p$ for some p has been considered for second order and powers of second order operators and for certain fourth order cases by Everitt and Giertz in their papers [3], [4] and [5].

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2. Statement of results. Define the class C_p by r is in C_p if r is locally L^p and $\sup_{0 \leq s < \infty} \int_s^{s+1} |r|^p < \infty$. We can now state our main result.

THEOREM 1. *Let $1 < p < \infty$. If $q_i = b_i + r_i$ where b_i is essentially bounded and $r_i \in C_p$, $0 \leq i \leq n-1$, then M is separated in L^p .*

As a consequence of Theorem 1 we will obtain

THEOREM 2. *Suppose $1 < p < \infty$. If $q_i = b_i + s_i$ where b_i is essentially bounded and $s_i \in L^{p_i}$ with $p_i \geq p$ for $0 \leq i \leq n-1$, then M is separated in L^p .*

COROLLARY 1. *If $q \in L^{p_1}$ with $1 < p_1 < \infty$ and b is bounded, then $y^{(n)} + (b + q)y$ is separated in L^p for any p , $1 < p \leq p_1$.*

The case $n = 2$, $p = p_1 = 2$ is part IV of Theorem 2 in [2]. (Their result is stated only for $b \equiv 0$ —but the extension to general bounded b follows easily.)

DEFINITION. Let $D(M, p)$ denote the linear manifold of all functions $y \in L^p$ satisfying $y^{(n-1)}$ is absolutely continuous and $My \in L^p$. In other words $D(M, p)$ is the domain of the maximal operator associated with the expression M in the space L^p —for a discussion of maximal operators in the context of the L^p spaces the reader is referred to the book by Goldberg [6].

Clearly if M is separated in L^p then $y^{(n)} \in L^p$ for any $y \in D(M, p)$. It follows from a well-known a priori bound—to be listed as Lemma 1 below—that $y \in L^p$, $y^{(n-1)}$ absolutely continuous and $y^{(n)} \in L^p$ and $1 \leq p \leq \infty$ imply that $y^{(i)} \in L^p$ for all i , $0 \leq i \leq n$. This establishes

THEOREM 3. *If M is L^p separated for $1 \leq p \leq \infty$, then $y^{(i)} \in L^p$ for all i , $0 \leq i \leq n$, and any $y \in D(M, p)$.*

As a consequence of Theorems 1 and 3 we obtain

COROLLARY 2. *If, for some p , $1 < p < \infty$, the coefficients q_i satisfy the hypotheses of Theorems 1 or 2, then $y^{(i)} \in L^p$ for $0 \leq i \leq n$ and any $y \in D(M, p)$. In particular if y is an L^p solution of $My = 0$, then $y^{(i)} \in L^p$ for $0 \leq i \leq n$ (given that the coefficients $q_i = b_i + r_i$ with b_i bounded and r_i either in C_p or in L^{p_i} for $p_i \geq p$).*

The last statement with $r_i \equiv 0$ for $0 \leq i \leq n-1$ and $p = 2$ reduces to Lemma 34 on p. 1428 in volume II of the book by Dunford and Schwartz [1].

3. Proofs, lemmas and a remark. Our proofs are based on three known results. We state these results only for the cases we need.

The first lemma is contained in Lemma VI.6.1, p. 157 of [6]. The case $p = 2$ is due to I. Halperin [7].

LEMMA 1. *Let $1 \leq p \leq \infty$. If $y \in L^p$ and $y^{(n-1)}$ is absolutely continuous and $y^{(n)} \in L^p$, then all the derivatives $y^{(i)}$, $0 \leq i \leq n$, are in L^p .*

Consider $Ty = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y$ with a_i essentially bounded, $0 \leq i \leq n-1$.

LEMMA 2. (See Theorem VI.6.2, p. 160 in [6].) *Let $1 \leq p \leq \infty$. If $y \in D(T, p)$, then $y^{(i)} \in L^p$ for $0 \leq i \leq n$.*

Consider an $(n - 1)$ th order operator B defined by

$$By = c_{n-1}y^{(n-1)} + \cdots + c_1y' + c_0$$

with each c_i in class C_p .

The next lemma is contained in Theorem VI. 8.1, p. 166 of [6]; its proof is on p. 169.

LEMMA 3. Let $1 < p < \infty$. (a) If $y \in D(T + B, p)$, then $y \in D(T, p)$. (b) Given $\varepsilon > 0$, there exists a constant K such that for any $y \in D(T, p)$

$$\|c_k y^{(k)}\|_p \leq \varepsilon \|y^{(n)}\|_p + K \|y\|_p \quad \text{for } 0 \leq k \leq n - 1.$$

(Here $\|f\|_p$ denotes the L^p norm of the function f .)

We can now proceed with the proof of Theorem 1:

Let $a_i = b_i$ and $c_i = r_i$ for $i = 0, 1, \dots, n - 1$. Then $My = Ty + By$. Now suppose $y \in D(M, p) = D(T + B, p)$. Then $y \in D(T, p)$ by (a) of Lemma 3. Hence $y^{(n)} \in L^p$ by Lemma 2. Part (b) of Lemma 3 then implies $r_i y^{(i)} \in L^p$ for $i = 0, \dots, n - 1$. Since by Lemma 2, $y^{(i)} \in L^p$ and b_i is bounded, it follows that $q_i y^{(i)} = b_i y^{(i)} + r_i y^{(i)} \in L^p$ for $0 \leq i \leq n - 1$. Therefore M is separated and the proof of Theorem 1 is complete.

Theorem 2 follows from Theorem 1 and the fact that an L^{p_i} function is the sum of a bounded function and a function in L^p for $p_i \geq p$.

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