THE EXACT CARDINALITY OF THE SET OF INVARIANT MEANS ON A GROUP

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ABSTRACT. The purpose of this note is to show that if G is an infinite amenable group then G has exactly $2^{|G|}$ invariant means where $|G|$ denotes the cardinality of G .

Let G be a discrete group, $m(G)$ the space of bounded real functions on G with the sup norm. For $x \in G$ and $f \in m(G)$, $l_x f$ and $r_x f$ are defined by $(l_x f)(y) = f(xy)$ and $(r_x f)(y) = f(yx)$, respectively, $y \in G$. $\mu \in m(G)^*$ is called a left (right) invariant mean on $m(G)$ if $\mu \geq 0$, $\|\mu\| = 1$ and $\mu(l_x f) = \mu(f)$ ($\mu(r_x f) = \mu(f)$) for $x \in G$ and $f \in m(G)$. We shall denote the set of left (right) invariant means on G by $ML(G)$ ($MR(G)$). Let $M(G) = ML(G) \cap MR(G)$, the set of two-sided invariant means, and $M^\sim(G) = \{\mu \in M(G) : \mu \text{ is inversion invariant}\}$. ($\mu \in m(G)^*$ is said to be inversion invariant if $\mu(f) = \mu(\tilde{f})$ for each $f \in m(G)$ where $\tilde{f}$ is defined by $\tilde{f}(x) = f(x^{-1})$.) G is said to be amenable if there is at least one left invariant mean on $m(G)$. When G is amenable, $M^\sim(G) \neq \emptyset$, cf. [3].

If X is a set, $|X|$ will denote the cardinality of X . If G is a finite group then $m(G)$ has a unique left invariant mean μ : $\mu(f) = (1/|G|)\sum\{f(x) : x \in G\}$. E. Granirer [5] proved that this is the only case that $ML(G)$ is a singleton. He actually proved that if G is infinite amenable then the vector space spanned by $ML(G)$ is infinite dimensional. In [1] we showed that if G is infinite amenable then $|ML(G)| \geq 2^c$ where c is the cardinality of the continuum. It implies that the linear span of $ML(G)$ is not even norm separable.

On the other hand, we are unable to find a theorem in the literature stating that $M(G)$ is not a singleton if G is infinite. But if G is assumed to be *countably infinite* and amenable then we do know that $|M(G)| \geq 2^c$ [2, Theorem 4.3].

Note that if G is infinite then $|m(G)^*| = 2^{|G|}$. Since $ML(G) \subset m(G)^*$, $|ML(G)| \leq 2^{|G|}$. Our main result is

THEOREM 1. *Let G be an infinite amenable group. Then $|ML(G)| = |M(G)| = |M^\sim(G)| = 2^{|G|}$.*

Clearly it is enough to show that $|M^\sim(G)| \geq 2^{|G|}$. In [2, Theorem 4.3] it is

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proved that if G is countably infinite then there is a one-one mapping of a set $\mathfrak{N}$ of cardinality $2^c = 2^{2^{\aleph_0}}$ into $M(G)$. The fact that the sequence $\{U_n\}$ in [2] can be chosen to be symmetric implies that the image of $\mathfrak{N}$ is actually contained in $M^*(G)$; see [2, pp. 450–451]. Therefore it remains to consider the case $|G| > \aleph_0$. We shall actually prove the stronger

THEOREM 2. *Let G be an uncountable amenable group. Then there is a family of subsets $\{E_\theta\}_{\theta \in \Theta}$ of G , $|\Theta| = 2^{|G|}$, such that each set function on $\{E_\theta\}_{\theta \in \Theta}$ with values in $[0, 1]$ is the restriction of an element of $M^*(G)$.*

Since there are $2^{2^{|G|}}$ such set functions, the uncountable case of Theorem 1 follows directly from the above theorem. To prove Theorem 2 we shall follow closely the steps of the proof of the well-known theorem of Kakutani and Oxtoby [7]. Their theorem is also reproduced in full in Hewitt and Ross [6, §16]. Therefore we shall skip some of the details. Our notation is similar to that of [6].

From now on G will denote a fixed amenable group with $|G| > \aleph_0$. Let T be the set of mappings $x \rightarrow ax$, $x \rightarrow xa$, $x \rightarrow x^{-1}$ of G onto itself ($a \in G$). Note that $|T| = |G|$. A set $X \subset G$ is said to be *almost invariant* if $|\tau X \triangle X| < |G|$ for each $\tau \in T$. The family of almost invariant subsets of G forms an algebra.

LEMMA 1. *There exists a family of subsets $\{X_\nu\}_{\nu \in P}$ of subsets of G such that:*

- (i) $|P| = |G|$;
- (ii) the sets X_ν are mutually disjoint;
- (iii) $\cup \{X_\nu: \nu \in P_0\}$ is almost invariant for each subset P_0 of P ;
- (iv) $|X_\nu| = |G|$ for each $\nu \in P$.

PROOF. Let ω be the first ordinal number with cardinality $|G|$.

Let $\{\tau_\alpha\}_{1 \leq \alpha < \omega}$ be a well ordering of T . For $1 \leq \alpha < \omega$, $x \in G$ set $C_\alpha(x) = \{\tau_{\beta_1}^{\epsilon_1} \circ \dots \circ \tau_{\beta_n}^{\epsilon_n} x: 1 \leq \beta_k \leq \alpha, \epsilon_k = 1 \text{ or } -1, k = 1, 2, \dots, n, \text{ and } n = 1, 2, \dots\}$. Note that $x \in C_\alpha(x)$; if $\beta \leq \alpha$ then $\tau_\beta(C_\alpha(x)) = C_\alpha(x)$ and $|C_\alpha(x)| < |G|$. (The last inequality holds since $|G| > \aleph_0$.) A transfinite double sequence X_β^α , $1 \leq \beta \leq \alpha < \omega$, in G can be constructed such that the sets $C_\alpha(x_\beta^\alpha)$, $1 \leq \beta \leq \alpha < \omega$, are mutually disjoint. Let $P = \{\nu: \nu \text{ ordinal number, } 1 \leq \nu < \omega\}$ and set $X_\nu = \cup \{C_\alpha(x_\nu^\alpha): \nu \leq \alpha < \omega\}$. Then $\{X_\nu\}_{\nu \in P}$ is what we want. The details can be found on pp. 218–219 of [6].

LEMMA 2. *There exists a collection of subsets of P , $\{P_\theta\}_{\theta \in \Theta}$ such that:*

- (i) $|\Theta| = 2^{|G|}$,
- (ii) $\cap_{k=1}^n P_{\theta_k}^{\epsilon_k} \neq \emptyset$ for each finite sequence $\theta_1, \dots, \theta_n$ of distinct elements of Θ and for $\epsilon_k = 1$ or $'$. (If $A \subset P$, $A^1 = A$, $A' = P \setminus A$.)

In [6, p. 220] the above lemma is proved for the case that $|P| = |G| = c$. Their proof also works here.

PROOF OF THEOREM 2. For each $\theta \in \Theta$, set $E_\theta = \cup \{X_\nu: \nu \in P_\theta\}$. Then, by Lemmas 1 and 2, for each sequence $\theta_1, \dots, \theta_n$ of distinct elements in Θ and for $\epsilon_k = 1$ or $'$, $\cap_{k=1}^n E_{\theta_k}^{\epsilon_k}$ is almost invariant and with cardinality equal to $|G|$. (Here $E_{\theta_k}' = G \setminus E_{\theta_k}$.) Note that for each finite sequence of distinct elements in Θ the collection $\{\cap_{k=1}^n E_{\theta_k}^{\epsilon_k}: \epsilon_k = 1 \text{ or } '\}$ forms a partition of G

into 2^n disjoint sets. Let $\mathfrak{G}$ be the family of sets consisting of the empty set and sets of the form

$$(1) \quad E = \bigcup \left\{ \bigcap_{k=1}^n E_{\theta_k}^{\epsilon_k} : (\epsilon_1, \dots, \epsilon_n) \in \Gamma \right\}$$

where Γ is a subset of $\{1, '\}^n$, $\theta_1, \dots, \theta_n$ is a sequence of distinct elements in Θ and $n = 1, 2, \dots$. Suppose E is as in (1) and $\theta_1, \dots, \theta_n, \theta_{n+1}, \dots, \theta_m$ is a sequence of distinct elements from Θ . Then E can also be written as

$$(2) \quad E = \bigcup \left\{ \bigcap_{k=1}^m E_{\theta_k}^{\epsilon_k} : (\epsilon_1, \dots, \epsilon_n, \dots, \epsilon_m) \in \Gamma^- \right\}$$

where $\Gamma^- \subset \{1, '\}^m$ is defined by $(\epsilon_1, \dots, \epsilon_m) \in \Gamma^-$ if and only if $(\epsilon_1, \dots, \epsilon_n) \in \Gamma$. Using this remark it is easy to see that $\mathfrak{G}$ is an algebra (of almost invariant subsets of G).

Denote the set of all functions from Θ to the closed interval $[0, 1]$ by Z . Note that $|Z| = 2^{2^{|\Theta|}}$. For each $\xi \in Z$ set $\phi_\xi(E_\theta) = \xi(\theta)$ and $\phi_\xi(E_\theta') = 1 - \xi(\theta)$. Extend ϕ_ξ to $\mathfrak{G}$ as follows: $\phi_\xi(\emptyset) = 0$, $\phi_\xi(E) = \sum \{ \prod_{k=1}^n \phi_\xi(E_{\theta_k}^{\epsilon_k}) : (\epsilon_1, \dots, \epsilon_n) \in \Gamma \}$ if E is as in (1). By (2) it is easy to see that ϕ_ξ is a well-defined finitely additive measure on $\mathfrak{G}$ and $\phi_\xi(G) = 1$.

Let $\mathfrak{F} = \{F \subset G : |F \triangle E| < |G| \text{ for some } E \in \mathfrak{G}\}$. Using the fact that each $E \in \mathfrak{G}$ is almost invariant one sees:

(3) If $E \in \mathfrak{G}$ and $|F \triangle E| < |G|$ then $|\tau F \triangle E| < |G|$ for each $\tau \in T$; therefore $\mathfrak{F}$ is closed under T .

Extend ϕ_ξ to $\mathfrak{F}$ by setting $\phi_\xi(F) = \phi_\xi(E)$ if $|F \triangle E| < |G|$, $E \in \mathfrak{G}$. ϕ_ξ is well defined on $\mathfrak{F}$ since if $|F \triangle E_1| < |G|$, $|F \triangle E_2| < |G|$, $E_i \in \mathfrak{G}$, then $|E_1 \triangle E_2| < |G|$ and hence $E_1 = E_2$ by (2), Lemma 2(ii) and the definition of E_θ . ϕ_ξ is clearly finitely additive on $\mathfrak{F}$ and, by (3), if $F \in \mathfrak{F}$ and $\tau \in T$, $\phi_\xi(\tau F) = \phi_\xi(F)$.

Let $J = \{ \sum_{i=1}^k c_i \chi_{F_i} : F_i \in \mathfrak{F}, c_i \in R \text{ and } k = 1, 2, \dots \}$ be the vector subspace of $m(G)$ spanned by the characteristic functions χ_F , $F \in \mathfrak{F}$. If $f \in J$, $x \in G$ then $l_x f, r_x f, \tilde{f}$ belong to J . Extend ϕ_ξ to J in the natural way: $\phi_\xi(\sum_{i=1}^k c_i \chi_{F_i}) = \sum_{i=1}^k c_i \phi_\xi(F_i)$. Then ϕ_ξ is a two-sided invariant mean on J which is also inversion invariant.

Now set $K = \{ \lambda : \lambda \in m(G)^*, \|\lambda\| = 1, \lambda \geq 0, \lambda|J = \phi_\xi \}$. Then K is a nonempty ω^* -compact convex subset of $m(G)^*$ and $l_x^* K \subset K$, $r_x^* K \subset K$ for each $x \in G$. By Day's fixed point theorem [4] there exist $\lambda'_\xi, \lambda''_\xi \in K$ such that $l_x^* \lambda'_\xi = \lambda'_\xi$ and $r_x^* \lambda''_\xi = \lambda''_\xi$, i.e., $\lambda'_\xi \in ML(G)$, $\lambda''_\xi \in MR(G)$ and $\lambda'_\xi|J = \phi_\xi$, $\lambda''_\xi|J = \phi_\xi$. (Note that this is the only place the fact that G is amenable is used.) Define λ_ξ as follows: $\lambda_\xi(f) = \lambda'_\xi(f')$ where $f'(x) = \lambda''_\xi(l_x f)$ ($x \in G$). Then $\lambda_\xi \in M(G)$ and $\lambda_\xi|J = \phi_\xi$. Finally set $\mu_\xi(f) = \frac{1}{2} \lambda_\xi(f + \tilde{f})$, $f \in m(G)$. Then $\mu_\xi \in M(G)$, $\mu_\xi|J = \phi_\xi$ and, hence, the proof of Theorem 2 is completed.

REMARKS. 1. The following shorter and more direct proof of Theorem 1 is provided by the referee. Let G be as in Theorem 2. Construct $\{X_\nu\}_{\nu \in P}$ as in Lemma 1. Let η be a finitely additive probability measure on the power sets of P . (Since η can be considered as a probability measure on βP , the

Stone-Čech compactification of the discrete set P , one sees that there are $2^{2^{|G|}}$ such η 's.) Let $\mathfrak{F}$ be the family of all sets of the form $E_Q = \bigcup \{X_\nu : \nu \in Q\}$, $Q \subset P$. Without loss of generality one may assume that $\bigcup \{X_\nu : \nu \in P\} = G$ and, hence, $\mathfrak{F}$ is a σ -algebra of almost invariant subsets of G . Now set $\phi(E_Q) = \eta(Q)$. Extend ϕ to an element in $M^*(G)$ as in the proof of Theorem 2.²

2. In a countably infinite group a set A is almost invariant if and only if $\tau A \triangle A$ is finite for each $\tau \in T$. In the additive group of integers $\mathbf{Z}$ an almost invariant set is either finite or is a complement of a finite set. Therefore Lemma 1 does not hold for $\mathbf{Z}$. On the other hand, it is not hard to see that if G is a countably infinite locally finite group then Lemma 1 holds, e.g., [8, p. 19].

3. For a group G , each $\tau \in T$ can be extended to a homeomorphism of βG onto βG which again will be denoted by τ . A set $K \subset \beta G$ is said to be invariant if $\tau K \subset K$ for each $\tau \in T$. Joe Rosenblatt has recently communicated to us the following fact: If G is an uncountable group, then using the existence of $\{E_\theta\}_{\theta \in \Theta}$ it can be shown that βG contains $2^{2^{|G|}}$ mutually disjoint closed invariant sets. If, in addition, G is amenable, then there exist $2^{2^{|G|}}$ elements in $M^*(G)$ with mutually disjoint supports. The latter fact also follows from Theorem 2. Indeed if $\xi, \zeta: \Theta \rightarrow \{0,1\}$, $\xi \neq \eta$, then there exists E_θ such that $\mu_\xi(E_\theta) \neq \mu_\zeta(E_\theta)$ and hence $\text{support } \mu_\xi \cap \text{support } \mu_\zeta = \emptyset$. We are unable to decide whether there are 2^c mutually singular elements in $M^*(G)$ if G is countably infinite. In [1] we are able to show that $ML(G)$ contains 2^c mutually singular elements.

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