ARITHMETIC MEANS OF FOURIER COEFFICIENTS

RAJENDRA SINHA

ABSTRACT. Given the Fourier coefficients of an even continuous function, we find a necessary and sufficient condition such that their arithmetic means are the Fourier coefficients of an odd continuous function. A similar result is shown for those Lipschitz classes whose elements are automatically equivalent to continuous functions.

Introduction. Let $\sum_{n=1}^{\infty} a_n \cos nx$ and $\sum_{n=1}^{\infty} b_n \sin nx$ be the Fourier series of f_c and f_s , respectively. Hardy showed that $\sum_{n=1}^{\infty} [(a_1 + \cdots + a_n)/n] \cos nx$ and $\sum_{n=1}^{\infty} [(b_1 + \cdots + b_n)/n] \sin nx$ [2, p. 51] will also be Fourier series—let $T_H f_c$ and $T_H f_s$, respectively, represent these two series. It was shown by M. Kinukawa and S. Igari [5, Theorem 2] that $f_c \in L^\infty \Rightarrow T_H \tilde{f}_c \in L^\infty$. In this paper we show that if $f_c \in C$, then $f_c(0) = 0 \Leftrightarrow T_H \tilde{f}_c \in C$.

A. Konyushkov [6, Theorem 15] showed that for $0 < \alpha < 1/p \leq 1$, $f_c \in \Lambda_\alpha^p \Rightarrow T_H f_c \in \Lambda_\alpha^p$. In this paper we consider the case $0 < 1/p < \alpha \leq 1$, and show that if $f_c \in \Lambda_\alpha^p$, then $\sum_{n=1}^{\infty} a_n = 0 \Leftrightarrow T_H \tilde{f}_c \in \Lambda_\alpha^p$.

All the notations, unless otherwise mentioned, are taken from [9]. Every function is supposed to be defined a.e., periodic with period 2π and integrable on $[-\pi, \pi]$. We shall not distinguish between equivalent functions. Given a function f , by $f(t)$ we would mean $\lim_{n \rightarrow \infty} \sigma_n(t; f)$ —whenever it exists. For convenience we shall write the Lipschitz classes Λ_α^p and Λ_α as $\Lambda(\alpha, p)$ and $\Lambda(\alpha)$, respectively.

Definition. For f and $\tilde{f} \in L^1$, we define $T_S f$ as the function representing the Fourier series $\sum_{n=1}^{\infty} [S_n(0; f)/n] \sin nt$.

We can easily check the following:

- (i) $T_S f_c = T_H \tilde{f}_c$.
- (ii) For $0 < \alpha \leq 1 \leq p < \infty$ and $\alpha p \neq 1$,

$$T_S f_c \in \Lambda(\alpha, p) \Leftrightarrow T_H f_c \in \Lambda(\alpha, p).$$
- (iii) For $0 < 1/p < \alpha \leq 1$, by [4, Theorem 5 (ii)],

$$f_c \in \Lambda(\alpha, p) \Rightarrow \sum_{n=1}^{\infty} a_n = \lim_{n \rightarrow \infty} \sigma_n(0; f) = f(0).$$

Now we state our results in the following form.

THEOREM 1. *Let $f \in C$. Then $f(0) = 0 \Leftrightarrow T_S f \in C$.*

Received by the editors January 24, 1975.

AMS (MOS) subject classifications (1970). Primary 42A16; Secondary 26A16.

Key words and phrases. Fourier series, conjugate function, Lipschitz class, strong summability, closed graph theorem.

© American Mathematical Society 1976

THEOREM 2. Let $f \in \Lambda(\alpha, p)$ where $0 < 1/p < \alpha \leq 1$. Then $f(0) = 0 \Leftrightarrow T_S f \in \Lambda(\alpha, p)$.

LEMMA 1.1. For $1 < p \leq \infty$, $\|T_S f\|_p \leq A_p \|f\|_p$ where A_p is a constant depending upon p only.

PROOF. Let $f \sim \sum_{n=0}^{\infty} (a_n \cos nt + b_n \sin nt)$. Take f_c and f_s as before. Then

$$T_S f(t) = T_H \tilde{f}_c(t) + a_0(\operatorname{sgn} t)(\pi - |t|)/2 \quad \text{for a.e. } t.$$

Hence, for $1 < p < \infty$, by [2, p. 51],

$$f \in L^p \Rightarrow T_S f \in L^p.$$

Now using the analysis similar to that of [2], we can show that

$$(1.1) \quad T_H \tilde{f}_c(t) = \int_t^{\pi} \frac{\tilde{f}_c(u) du}{2 \tan u/2} + \beta(t) \quad \text{for a.e. } t > 0$$

where $\sum_{n=1}^{\infty} [a_n/2n] \sin nt = \beta(t) \in C$.

Hence, by [6, Theorem 2],

$$f \in L^{\infty} \Rightarrow f_c \in L^{\infty} \Rightarrow T_H \tilde{f}_c \in L^{\infty} \Rightarrow T_S f \in L^{\infty}.$$

Now using the closed graph theorem, we can easily show the continuity of the operator T_S .

PROOF OF THEOREM 1. ($\Rightarrow$) Let $\sigma_n f(x) \equiv \sigma_n(x; f)$. Define

$$e_n(x) \equiv \sigma_n f(x) - [\sigma_n f(0)] \cos(n+1)x.$$

Then

$$\|e_n - f\|_{\infty} \leq \|\sigma_n f - f\|_{\infty} + |\sigma_n f(0)|.$$

But $f(0) = 0 \Rightarrow |\sigma_n f(0)| \rightarrow 0$ as $n \rightarrow \infty$. Therefore $\|e_n - f\|_{\infty} \rightarrow 0$, and, hence, $\|T_S e_n - T_S f\|_{\infty} \rightarrow 0$ as $n \rightarrow \infty$. But $T_S e_n$ is continuous for all n , being a polynomial of degree n . Thus $T_S f \in C$.

($\Rightarrow$) Take $F = f - f(0)$. Then

$$T_S F(t) = T_S f(t) - f(0)(\operatorname{sgn} t)(\pi - |t|)/2 \quad \text{for a.e. } t.$$

But $T_S f$ and $T_S F$ are continuous functions—the second one is continuous because $F(0) = 0$. Hence, $f(0) = 0$. Q.E.D.

REMARK. Once $T_S f$ is continuous, it is not very hard to show that the Fourier series of $T_S f$ converges uniformly. We just have to consider the difference of Cesàro means and the partial sums of $T_S f$ and use the strong summability of Fourier series of f at 0.

COROLLARY 1.1. Let f be a continuous function and a_0 the constant term of its Fourier expansion. Then for every x ,

$$\lim_{t \rightarrow 0+} \frac{1}{\pi} \int_t^{\pi} \frac{\tilde{f}(x+u) - \tilde{f}(x-u)}{2 \tan u/2} du = f(x) - a_0.$$

PROOF. It is enough to prove this for $x = 0$. We can also assume that $f = f_c$. Take $G = f_c - f_c(0)$. Then, by (1.1),

$$T_S G(t) = \int_t^\pi \frac{\tilde{f}_c(u)}{2 \tan u/2} du + F(t) - f_c(0)(\pi - t)/2 \quad \text{for } \pi \geq t > 0$$

and

$$T_S G(-t) = -T_S G(t) \quad \text{for } \pi \geq t > 0.$$

But $T_S G \in C$ by the theorem. Hence,

$$\lim_{t \rightarrow 0+} \frac{1}{\pi} \int_t^\pi \frac{\tilde{f}_c(u)}{2 \tan u/2} du = f_c(0). \text{ Q.E.D.}$$

PROOF OF THEOREM 2. ($\Leftarrow$) Since f and $T_S f$ are continuous by [4, Theorem 5], the necessity of the condition is obvious from Theorem 1.

($\Rightarrow$) We shall consider two cases.

Case 1. $f \in \Lambda(1, p)$ where $1 < p < \infty$. By [3, Theorem 24], $f(x) - f(0) = \int_0^x F(t) dt$ where $F \in L^p$. Let $f \sim \sum_{n=0}^\infty (a_n \cos nt + b_n \sin nt)$. Then

$$F(t) = f'(t) \sim \sum_{n=1}^\infty (-na_n \sin nt + nb_n \cos nt).$$

So

$$\sum_{n=1}^\infty na_n \sin nt \sim [F(-t) - F(t)]/2 \in L^p.$$

Hence, by [1, Theorem 2],

$$\sum_{n=1}^\infty \left(\sum_{j=n}^\infty a_j \right) \cos nt \sim F^* \in L^p.$$

But $\sum_{j=0}^\infty a_j = f(0) = 0$. Hence

$$-\sum_{n=1}^\infty S_n(0; f) \cos nt \sim F^* \in L^p.$$

Now, by [3, Theorem 22], $T_S f \in \Lambda(1, p)$.

Case 2. $f \in \Lambda(\alpha, p)$ where $0 < 1/p < \alpha < 1$. Define $\sigma_n f$ as before and $\sigma_{n,j}$ by

$$\sigma_{n,j}(x) = \sum_{i=0}^j [1 - i/(n+1)] A_i(x; f).$$

By [7, Theorem 6], $\|f - \sigma_n f\|_p = O(n^{-\alpha})$. Now $T_S \sigma_n f = P_n + Q_n$ where

$$P_n(t) = \sum_{j=1}^n \left[\frac{\sigma_{n,j}(0)}{j} \right] \sin jt \quad \text{and} \quad Q_n(t) \sim [\sigma_n f(0)] \sum_{j=n+1}^\infty \frac{(\sin jt)}{j}.$$

By [4, Theorem 5], $f \in \Lambda(\alpha - 1/p)$. Hence, by [9, Vol. I, p. 123], $|\sigma_n f(0)| = O(n^{-\alpha+1/p})$. Now

$$\sum_{j=1}^\infty (\sin jt)/j \sim (\operatorname{sgn} t)(\pi - |t|)/2 \in \Lambda(1/p, p).$$

Hence, by [7, Theorem 5], $\|Q_n/\sigma_n f(0)\|_p = O(n^{-1/p})$. Using Lemma 1.1 and the above relations, we get

$$\begin{aligned}\|T_S f - P_n\|_p &\leq \|T_S f - T_S \sigma_n f\|_p + \|T_S \sigma_n f - P_n\|_p \\ &\leq A_p \|f - \sigma_n f\|_p + \|Q_n\|_p = O(n^{-\alpha}).\end{aligned}$$

Hence, by [7, Theorem 2], $T_S f \in \Lambda(\alpha, p)$. Q.E.D.

REMARKS. (i) Using the above technique, we can easily show that $f \in \Lambda(\alpha)$ and $f(0) = 0 \Rightarrow T_S f \in \Lambda(\alpha)$ where $0 < \alpha < 1$.

(ii) Using Corollary 2.1 of [8], we can also show that $f \in \Lambda(1, 1) \Rightarrow T_S f \in \Lambda(1, 1)$. But the corresponding statement for the class $\Lambda(1/p, p)$, where $p > 1$, is not true as can be seen by taking $f \sim \sum_{n=1}^{\infty} (\cos nx)/n$. Imposing the condition $|\sigma_n f(0)| = O(1)$, and using the above technique, one can prove that $f \in \Lambda(1/p, p)$ implies $T_S f \in \Lambda(1/p, p)$.

REFERENCES

1. R. Bellman, *Note on a theorem of Hardy on Fourier constants*, Bull. Amer. Math. Soc. **50**(1944), 741–744. MR6, 125.
2. G. H. Hardy, *Notes on some points in the integral calculus*, Messenger Math. **58**(1929), 50–52.
3. G. H. Hardy and J. E. Littlewood, *Some properties of fractional integrals. I*, Math. Z. **27**(1928), 565–606.
4. ———, *A convergence criterion for Fourier series*, Math. Z. **28**(1928), 612–634.
5. M. Kinukawa and S. Igari, *Transformations of conjugate functions*, Tôhoku Math. J. (2) **13**(1961), 274–280. MR26 #2829.
6. A. A. Konyuškov, *On Lipschitz classes*, Izv. Akad. Nauk SSSR Ser. Mat. **21**(1957), 423–448. (Russian) MR19, 546.
7. E. S. Quade, *Trigonometric approximation in the mean*, Duke Math. J. **3**(1937), 529–543.
8. Rajendra Sinha, *On an infinite linear combination of partial sums of Fourier series*, Studia Math. (to appear).
9. A. Zygmund, *Trigonometric series*. Vols. I, II, 2nd ed., Cambridge Univ. Press, New York and London, 1968. MR38 #4882.

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907

Current address: Department of Mathematics, Birla Institute of Technology and Science, Pilani, Rajasthan, India 333031