ON REALIZING CENTRALIZERS OF CERTAIN ELEMENTS IN THE FUNDAMENTAL GROUP OF A 3-MANIFOLD

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ABSTRACT. The main result in this note is that if λ is a simple loop in the boundary of a compact, irreducible, orientable 3-manifold M and $[\lambda] \neq 1 \in \pi_1(M)$, one can represent the centralizer of $[\lambda]$ in $\pi_1(M)$ by a Seifert fibred submanifold of M .

Introduction. The main result in this note is a partial answer to a question of Jaco [2]. Jaco has shown in [2] that the centralizer of a nontrivial element in the fundamental group of a sufficiently large compact, orientable 3-manifold is isomorphic to the fundamental group of a Seifert fibre space. He also suggests that one might geometrically realize this group by a submanifold of the ambient manifold. It is the purpose of this note to show that the above realization can in fact be made if the element is represented by a simple loop in the boundary of the 3-manifold and the 3-manifold is irreducible.

Proposition 7.1 in [2] is quite similar to our Theorem 2. Our notation and definitions are standard unless otherwise indicated.

We say that a manifold N is *properly embedded* in a manifold M if $N \cap \partial M = \partial N$. Let A be an annulus. A *spanning arc* α of A is an arc properly embedded in A such that $A - \alpha$ is simply connected. Throughout the remainder of this paper α will denote a spanning arc of A .

PROPOSITION 1. *Let $A_1, \dots, A_m$ be a collection of annuli properly embedded in M such that $A_i \cap A_j = \partial A_i = \partial A_j$ for $1 \leq j < i \leq m$. Let $f: (A, \partial A) \rightarrow (M, \partial M)$ be a map such that (1) $f(\partial A) = \partial A_1$, (2) $f_*: \pi_1(A) \rightarrow \pi_1(M)$ is monic, and (3) $f(\alpha)$ is not homotopic rel its boundary to an arc in $\bigcup_{i=1}^m A_i$. Then there is an embedding $g: (A, \partial A) \rightarrow (M, \partial M)$ such that*

- (1) $g(A) \cap A_i = \partial A_i$ for $i = 1, \dots, m$,
- (2) $g_*: \pi_1(A) \rightarrow \pi_1(M)$ is monic,
- (3) $g(\alpha)$ is not homotopic rel its boundary to an arc in $\bigcup_{i=1}^m A_i$.

PROOF. We show first that $f|_{\partial A}$ may be assumed to be an embedding. Let C_1 and C_2 be the components of ∂A . Let $(\tilde{M}, \rho)$ be the covering space of M associated with the subgroup of $\pi_1(M)$ generated by the class of the simple loop $f(C_1)$. Since $f_*\pi_1(A) \subseteq \rho_*\pi_1(\tilde{M})$, there is a map $\tilde{f}: (A, \partial A) \rightarrow (\tilde{M}, \partial \tilde{M})$ such that $\rho\tilde{f} = f$. It is a consequence of the theorem in [8] that there is an embedding $\tilde{f}_1: (A, \partial A) \rightarrow (\tilde{M}, \partial \tilde{M})$ such that $\tilde{f}_1(\partial A) = \tilde{f}(\partial A)$. We may

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suppose that $\tilde{f}_1(\partial\alpha) = \tilde{f}(\partial\alpha)$. Since $\pi_1(\tilde{M})$ is generated by $\tilde{f}_1 \cdot [C_1]$, we may suppose that the loop formed by traversing first $\tilde{f}_1(\alpha)$ and then $\tilde{f}(\alpha)$ is nullhomotopic.

Since the path $f(\alpha)$ is end-point fixed homotopic in M to the path $\rho\tilde{f}_1(\alpha)$, one may assume that $f|_{C_1}$ is an embedding of C_1 into ∂A_1 . Using a similar argument, we may show that there is no loss in generality in assuming further that $f|_{C_2}$ is also an embedding of C_2 into ∂A_1 .

It is a consequence of Theorem 1' in [1] that there is an embedding $g: (A, \partial A) \rightarrow (M, \partial M)$ such that $g(\partial A) = \partial A_1$ and $g(\alpha)$ is not homotopic rel its boundary to an arc in $\bigcup_{i=1}^n A_i$. After the usual general position and cutting arguments, we may suppose that $g^{-1}(\bigcup_{i=1}^m A_i)$ is the union of a collection of disjoint simple essential loops $\lambda_1, \dots, \lambda_r$. Let $\bar{A}_1, \dots, \bar{A}_{r-1}$ be the closures of the components of $A - \bigcup_{i=1}^r \lambda_i$. We may suppose that α meets each of the $\bar{A}_i$ in a spanning arc α_i for $i = 1, \dots, r-1$. Since $g(\alpha)$ is not homotopic rel its boundary to an arc on $\bigcup_{i=1}^m A_i$, there is a j such that $g(\alpha_j)$ is not homotopic rel its boundary to an arc on $\bigcup_{i=1}^m A_i$. Now it is easy to obtain the desired embedding by considering $g|_{\bar{A}_j}$. This completes the proof of Proposition 1.

Let G be a group and $\sigma \in G$. We denote the centralizer of σ in G , i.e. $\{g \in G: \sigma g = g\sigma\}$ by $C(\sigma)$. The following theorem gives a positive partial answer to a question posed by Jaco in [2].

THEOREM 1. *Let M be a compact, connected, irreducible, orientable 3-manifold, λ a simple loop in ∂M which is not nullhomotopic in M , and y a point in λ . Then there is a submanifold $N \subseteq M$ such that*

- (1) $\pi_1(N, y) = C([\lambda]) \subseteq \pi_1(M, y)$;
- (2) N is a (possibly trivial) Seifert fibre space;
- (3) $\partial N \cap \partial M$ is an annular neighborhood A^* of λ ;
- (4) $\partial N - A^*$ is incompressible in M .

PROOF. It is a consequence of the loop theorem [4], [6] that $[\lambda]$ is of infinite order in $\pi_1(M, y)$. Let $\bar{A}$ be an annular neighborhood of λ in ∂M . Let $A_1, \dots, A_n$ be a maximal collection of annuli properly embedded in M such that

- (1) $\partial A_i = \partial \bar{A}$ for $i = 1, \dots, n$.
- (2) $A_i \cap A_j = \partial A_i$ for $1 \leq i < j \leq n$.
- (3) If α_j is a spanning arc of A_j where $1 \leq j \leq n$, α_j is not homotopic rel its boundary to an arc in $\bigcup_{i \neq j}^n A_i \cup \bar{A}$.

It follows from the theorem on p. 60 in [7] that there are at most finitely many disjoint nonparallel annuli properly embedded in M and satisfying condition (1) above.

Suppose $x \in C([\lambda])$. Then we claim x has a representative loop on $\bigcup_{i=1}^n A_i \cup \bar{A}$.

If our claim is false, we let T be a torus and λ_1 and λ_2 simple loops on T such that $\lambda_1 \cap \lambda_2$ is a single point and $T - (\lambda_1 \cup \lambda_2)$ is simply connected. Since x and $[\lambda]$ commute, there is a map $\tilde{f}: T \rightarrow M$ carrying a neighborhood R of λ_1 homeomorphically onto $\bar{A}$ and λ_2 to a representative of x in $\pi_1(M)$. One obtains an annulus A by removing the interior of R from T so that $\tilde{f}$ induces a map $f: (A, \partial A) \rightarrow (M, \partial \bar{A})$. We observe that if α is a spanning arc of A , $f(\alpha)$ is not homotopic rel its boundary to an arc on $\bigcup_{i=1}^n A_i \cup \bar{A}$, since x

has no representative in that set. Thus it follows from Proposition 1 that the collection was not chosen to be maximal. Our claim follows.

Let $X = \bar{A} \cup \bigcup_{i=1}^n A_i$. We claim that $\pi_1(X)$ is the direct product of a free group with the integers. This can be seen by

- (1) choosing a spanning arc $\bar{\alpha}$ of $\bar{A}$,
- (2) choosing a spanning arc α_i of A_i for $i = 1, \dots, n$ so $\partial\alpha_i = \bar{\alpha} \cap A_i$,
- (3) letting $X_0 = \bar{\alpha} \cup \bigcup_{i=1}^n \alpha_i$,
- (4) observing that X is naturally homeomorphic to $X_0 \times S^1$.

Let N_1 be a regular neighborhood of X in M . Note that by construction N_1 is homeomorphic to the product of a surface F with S^1 and that λ is the product of a point with S^1 in this structure. Note each component of ∂N_1 is a torus. Suppose T is a component of $\partial N_1 - \bar{A}$ such that $\ker(\pi_1(T) \rightarrow \pi_1(M)) \neq 1$. Then since M is irreducible, it is a consequence of the loop theorem [4], [6] that T bounds a solid torus in M . Let N be the union of N_1 with all such solid tori. Now N is a Seifert fibre space and $\ker(\pi_1(N) \rightarrow \pi_1(M)) = 1$. This completes the proof of Theorem 1.

THEOREM 2. *Let M be a compact, irreducible, orientable 3-manifold with incompressible boundary and λ a simple loop in ∂M such that $[\lambda] \neq 1 \in \pi_1(M)$. Then if there exist integers $n > k \geq 1$ and $x \in \pi_1(M)$ such that $x^n = [\lambda]^k$, there is a solid torus N embedded in M such that $N \cap \partial M$ is an annular neighborhood of λ and $\sigma \in \pi_1(N)$ where $\sigma^n = [\lambda]^k$.*

PROOF. It is a consequence of the loop theorem [4], [6] that $[\lambda]$ is of infinite order. Let M^* be the Seifert fibre space whose existence is guaranteed by Theorem 1. We suppose that M^* is the union of $F \times S^1$ with a collection of solid tori (regular neighborhoods of the exceptional fibres) $N_1, \dots, N_k$ in the interior of M^* , that there is a point $y \in F$ such that $\lambda = \{y\} \times S^1$, and $N_i \cap F \times S^1 = \partial N_i$ for $i = 1, \dots, k$. Let α_i be a simple arc properly embedded in F such that $\partial(\alpha_i \times S^1)$ is the union of λ and a simple loop in ∂N_i for $i = 1, \dots, k$. Let R_1 be a regular neighborhood of $\alpha_1 \times S^1 \cup N_1$ and let M_1 be the closure of $M^* - R$. Note that R_1 is a solid torus.

By van Kampen's theorem $\pi_1(M^*)$ is the free product of $\pi_1(M_1)$ and $\pi_1(R_1)$ with amalgamation over $\pi_1(M_1 \cap R_1) = \langle [\lambda] \rangle$. Note that any conjugate of x in $\pi_1(M^*)$ is an n th root of $[\lambda]^k$ since $[\lambda]$ commutes with all elements of $\pi_1(M^*)$. Now it is a consequence of Lemma 10 in [5] that we may suppose that x has length 1 since $\text{length}[\lambda]^k = 1$ and $\text{length}[\lambda]^k = \text{length } x^n = n \cdot (\text{length } x)$ if $(\text{length } x) > 1$ where x is assumed to be an element represented by a cyclically reduced word. Thus we may suppose that $x \in \pi_1(R_1)$ or $x \in \pi_1(M_1)$.

If $x \in \pi_1(R_1)$, we are finished. Otherwise we let M_1^* be the closure of $M^* - N_1$. Now $x^n = [\lambda]^k$ holds in $\pi_1(M_1^*)$ since it holds in $\pi_1(M_1)$. Furthermore M_1^* has one less singular fibre than M^* . Since the relation $x^n = [\lambda]^k$ holds in $\pi_1(F \times S^1)$ only if n divides k , the desired result follows by induction on the number of singular fibres.

In [5] Shalen points out that he has left open the following question: Does every encrusted singular curve have a crust homeomorphic to $\mathbb{P}^2 \times S^1$? Since an "encrusted curve" is special and any "special conjugacy class" can be represented by a power of a simple loop by Proposition 2 in [5], one is

easily able to answer the above question in the affirmative using Theorem 2 above. Jaco has also given an answer to this question in [2].

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