CHARACTERIZATIONS OF COMPLETELY HAUSDORFF-CLOSED SPACES

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ABSTRACT. This paper gives several characterizations of completely Hausdorff-closed spaces.

1. Introduction. Our primary interest is the investigation of completely Hausdorff-closed spaces. Current characterizations of completely Hausdorff-closed spaces are given in terms of completely Hausdorff open filterbases [1, p. 103]. In this paper we give characterizations of completely Hausdorff-closed spaces in terms of nets and arbitrary filterbases. These characterizations are obtained mainly through the introduction of a type of convergence for filterbases and nets that we call f -convergence.

Throughout, $\text{cl}(A)$ will denote the closure of a set A and $\text{Int}(A)$ will denote the interior of A . Also, we will use the letter R to denote the real line.

2. Preliminary definitions and theorems. Let $f: X \rightarrow R$ be a continuous real-valued function from a topological space X into the reals and let $p \in X$. If H is an open set in R containing $f(p)$, then we say that f and H is an ordered pair for p and denote this by $(f, H)_p$.

DEFINITION 2.1. Let X be a topological space and let $F = \{A_\alpha: \alpha \in \Delta\}$ be a filterbase in X . Then F f -converges to $p \in X$ ($F \rightarrow_f p$) if for each ordered pair $(f, H)_p$ there exists an $A_\alpha \in F$ such that $A_\alpha \subset f^{-1}(H)$. The filterbase F f -accumulates to $p \in X$ ($F \propto_f p$) if for each ordered pair $(f, H)_p$ and for each $A_\alpha \in F$, $A_\alpha \cap f^{-1}(H) \neq \emptyset$.

We note that a filterbase $F = \{A_\alpha: \alpha \in \Delta\}$ f -converges to $p \in X$ if and only if for each ordered pair $(f, H)_p$ there exists $A_\alpha \in F$ such that $f(A_\alpha) \subset H$. Also F f -accumulates to $p \in X$ if and only if for each ordered pair $(f, H)_p$ and for each $A_\alpha \in F$, $f(A_\alpha) \cap H \neq \emptyset$.

Convergence and accumulation of filterbases in the usual sense, of course, imply f -convergence and f -accumulation, respectively. However, the converses do not hold as the next example shows.

EXAMPLE 2.2. Let $X = [0, 6]$ have as a subbase the usual open sets together with the set $S = \{r: 1 < r < 5 \text{ and } r \text{ is rational}\}$. Let $F = \{A_n: n \in N\}$ be a filterbase in X (where $A_n = \{x: 3 < x < 3 + 1/n \text{ and } x \text{ is irrational}\}$) and let $p = 3$. The filterbase F does not converge or accumulate in the usual sense to p but $F \rightarrow_f p$ and $F \propto_f p$.

There are a number of theorems concerning f -convergence and f -accumulation whose statements parallel those of convergence and accumulation

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tion in the usual sense. We give a sample of some of these theorems but omit their straightforward proofs.

THEOREM 2.3. *In a topological space X the following properties hold:*

- (a) *If F is a filterbase in X such that F f -converges to $p \in X$, then F f -accumulates to p and, in completely Hausdorff spaces, at no point other than p .*
- (b) *Let F_1 and F_2 be two filterbases in X where F_2 is stronger than F_1 . Then F_1 f -accumulates to $p \in X$ if F_2 f -accumulates to p .*
- (c) *A filterbase F_1 f -accumulates to $p \in X$ if and only if there exists a filterbase F_2 stronger than F_1 such that F_2 f -converges to p .*
- (d) *A maximal filterbase M in X f -accumulates to $p \in X$ if and only if M f -converges to p .*
- (e) *X is completely Hausdorff if and only if each f -convergent filterbase in X f -converges to exactly one point.*
- (f) *If X is completely regular, then a filterbase F converges (accumulates) to $p \in X$ if and only if F f -converges (resp., f -accumulates) to p .*

DEFINITION 2.4. Let X be a topological space and let $\phi: D \rightarrow X$ be a net in X . Then ϕ f -converges to $p \in X$ ($\phi \rightarrow_f p$) if for each ordered pair $(f, H)_p$ there exists a $d \in D$ such that $\phi(T_d) \subset f^{-1}(H)$ (where $T_d = \{c \in D: d < c\}$). The net ϕ f -accumulates to $p \in X$ ($\phi \alpha_f p$) if for each ordered pair $(f, H)_p$ and for every $d \in D$, $\phi(T_d) \cap f^{-1}(H) \neq \emptyset$.

Of course, if $\phi: D \rightarrow X$ is a net in X , the family $F(\phi) = \{\phi(T_d): d \in D\}$ is a filterbase in X and it is routine to verify that:

- (a) $F(\phi) \rightarrow_f p \in X$ if and only if $\phi \rightarrow_f p$.
- (b) $F(\phi) \alpha_f p \in X$ if and only if $\phi \alpha_f p$.

Conversely, every filterbase F in X determines a net $\phi: D \rightarrow X$ such that:

- (a) $F \rightarrow_f p \in X$ if and only if $\phi \rightarrow_f p$.
- (b) $F \alpha_f p \in X$ if and only if $\phi \alpha_f p$.

The construction of such a net is the same as that of [2, p. 213].

We next state a few theorems concerning f -convergence for nets.

THEOREM 2.5. *In a topological space X the following properties hold.*

- (a) *If ϕ is a net in X such that ϕ f -converges to $p \in X$, then ϕ f -accumulates to p , and in completely Hausdorff spaces, at no point other than p .*
- (b) *A net ϕ f -accumulates to $p \in X$ if and only if there exists a subnet of ϕ f -converging to p .*
- (c) *A universal net ϕ f -accumulates to $p \in X$ if and only if ϕ f -converges to p .*

3. Filterbase and net characterizations of completely Hausdorff-closed spaces. An open filterbase F in X is completely Hausdorff if and only if for each $p \notin A(F)$ (where $A(F)$ denotes the set of accumulation points of F), there exists an open set U containing p , $V \in F$, and a continuous real-valued function $f: X \rightarrow R$ such that $f(U) = \{0\}$ and $f(V) = \{1\}$ [1, p. 103].

Let $A = \{U_\alpha: \alpha \in \Delta_1\}$ and $B = \{V_\beta: \beta \in \Delta_2\}$ be covers of a space X . Then B is a continuous refinement of A if and only if for each $V_\beta \in B$, there is a $U_\alpha \in A$ and a continuous real-valued function $f: X \rightarrow R$ such that $f(V_\beta) = \{0\}$ and $f(X - U_\alpha) = \{1\}$. An open cover is completely Hausdorff if and only if it has a continuous refinement [1, p. 103].

A completely Hausdorff space X is completely Hausdorff-closed if and only if X is a closed set in every completely Hausdorff space in which it can be embedded [1].

LEMMA 3.1. *Let $F = \{O_\alpha: \alpha \in \Delta\}$ be an open completely Hausdorff filterbase on X . Then $A(F) = A_f(F)$ (where $A_f(F)$ denotes the set of f -accumulation points of F).*

PROOF. Clearly we only need to show that $A_f(F) \subset A(F)$. Therefore, suppose that $p \notin A(F)$. Then there exists an open set U containing p , $O_\alpha \in F$, and a continuous $f: X \rightarrow R$ such that $f(U) = \{0\}$ and $f(O_\alpha) = \{1\}$. Let $H = \{x \in R: -\frac{1}{2} < x < \frac{1}{2}\}$. Then $f(O_\alpha) \cap H = \emptyset$ shows that $p \notin A_f(F)$. Therefore, we conclude that $A(F) = A_f(F)$.

LEMMA 3.2. *Let $F = \{A_\alpha: \alpha \in \Delta\}$ be a filterbase on X and let $p \in X$. Then F does not f -accumulate to p if and only if there exists an ordered pair $(f, H)_p$ and an $A_\alpha \in F$ such that $\text{cl}(f(A_\alpha)) \cap \text{cl}(H) = \emptyset$.*

PROOF. The straightforward proof follows from Definition 2.1 and the regularity of the reals.

THEOREM 3.3. *Let X be a completely Hausdorff space. Then the following are equivalent.*

- (a) X is completely Hausdorff-closed.
- (b) Each filterbase F in X f -accumulates to some point $p \in X$.
- (c) Each maximal filterbase M in X f -converges to some point $p \in X$.

PROOF. (a) implies (b). Suppose there exists a filterbase $F = \{A_\alpha: \alpha \in \Delta\}$ in X that does not f -accumulate in X . Then for each $x \in X$ there exists an ordered pair $(f_x, H_x)_x$ and some $A_{\alpha(x)} \in F$ such that $\text{cl}(f_x(A_{\alpha(x)})) \cap \text{cl}(H_x) = \emptyset$ according to Lemma 3.2. Since R is normal, there exists an open set $B_x \supset \text{cl}(f_x(A_{\alpha(x)}))$ such that $\text{cl}(B_x) \cap \text{cl}(H_x) = \emptyset$. Let $K_x = f^{-1}(\text{cl}(B_x))$ and choose an open set U_x containing x such that $f(U_x) \subset H_x$. Then $U_x \cap K_x = \emptyset$ and $\text{Int}(K_x) \supset A_{\alpha(x)}$. Now by Theorem 4.1 of [2, p. 146] there exists a continuous $g_x: R \rightarrow R$ with the property that $g_x(\text{cl}(H_x)) = \{0\}$ and $g_x(\text{cl}(B_x)) = \{1\}$. Consequently for each $x \in X$, there exist open sets U_x and $G_x = X - K_x$ containing x and a continuous real-valued function $h_x = g_x f_x$ with the property that $h_x(U_x) = \{0\}$ and $h_x(X - G_x) = \{1\}$. Therefore, $\{G_x: x \in X\}$ is a completely Hausdorff open cover of X . Since X is completely Hausdorff-closed, there exists a finite subcollection $\{G_{x_i}: i = 1, 2, \dots, n\}$ such that $\bigcup_{i=1}^n \text{cl}(G_{x_i}) = X$ according to Theorem 4.9 of [1, p. 103]. We note that for each i , $X - \text{cl}(G_{x_i}) = \text{Int}(K_{x_i})$. Therefore,

$$\emptyset = \bigcap_{i=1}^n (X - \text{cl}(G_{x_i})) = \bigcap_{i=1}^n \text{Int}(K_{x_i}) \supset \bigcap_{i=1}^n A_{\alpha(x_i)}$$

which is a contradiction. We conclude that F f -accumulates in X .

(b) implies (a). Let $F = \{O_\alpha: \alpha \in \Delta\}$ be an open completely Hausdorff filterbase on X . By Lemma 3.1 and hypothesis (b) we have that $A_f(F) = A(F) \neq \emptyset$. Therefore, X is completely Hausdorff-closed according to Theorem 4.9 of [1, p. 103].

(b) implies (c). Let M be a maximal filterbase on X . Then M f -accumulates to some point $p \in X$ and, hence, f -converges to p by Theorem 2.3(d).

(c) implies (b). Let F be a filterbase in X . Then there exists a maximal filterbase M in X which is stronger than F . Since M f -converges to some point $p \in X$, F f -accumulates to p according to Theorem 2.3(b),

Our discussion in the previous section showed that filterbases and nets are "equivalent" in the sense of f -convergence and f -accumulation. Thus, we can now characterize completely Hausdorff-closed spaces in terms of nets.

THEOREM 3.4. *In a completely Hausdorff space the following are equivalent:*

- (a) X is completely Hausdorff-closed.
- (b) Each net in X has an f -accumulation point.
- (c) Each universal net f -converges.

REMARK 3.5. For each topological space (X, T) there is a corresponding topological space (X, T_*) called the semiregular space associated with (X, T) [1, p. 96]. The topology T_* is generated by the regular open sets in (X, T) . We note that when a space Y is regular, a map $f: (X, T) \rightarrow Y$ is continuous if and only if $f: (X, T_*) \rightarrow Y$ is continuous. Consequently, it follows that a space (X, T) is completely Hausdorff if and only if (X, T_*) is completely Hausdorff. Also, it is easy to see that a filterbase F on X f -accumulates to a point p in (X, T) if and only if F f -accumulates to p in (X, T_*) . With this in consideration we give the following theorem.

THEOREM 3.6. *A space (X, T) is completely Hausdorff-closed if and only if (X, T_*) is completely Hausdorff-closed.*

PROOF. The result follows from Theorem 3.3 and Remark 3.5.

4. First countable completely Hausdorff spaces. A space (X, T) is called first countable and minimal completely Hausdorff if T is first countable and completely Hausdorff, and if no first countable topology on X which is strictly weaker than T is completely Hausdorff. (X, T) is first countable and completely Hausdorff-closed if T is first countable and completely Hausdorff, and (X, T) is a closed subspace of every first countable completely Hausdorff space in which it can be embedded [1].

THEOREM 4.1. *A first countable completely Hausdorff space X is first countable and completely Hausdorff-closed if each countable filterbase on X f -accumulates to some point $p \in X$.*

PROOF. Assume that X is not completely Hausdorff-closed. Then there exists a first countable completely Hausdorff space Y and a homeomorphism $h: X \rightarrow h(X) \subset Y$ such that $h(X)$ is not closed in Y . Thus there exists a point $p \in Y - h(X)$ such that $p \in \text{cl}(h(X))$. Let $\{U_i: i = 1, 2, 3, \dots\}$ be a countable base at p . Then $F = \{U_i \cap h(X): i = 1, 2, \dots\}$ forms a countable open filterbase on $h(X)$. Since $h(X)$ is homeomorphic to X , F f -accumulates to some point $z \in h(X)$. It is clear that F converges to p . Therefore, by Theorem 2.3(a), p is equal to z which is a contradiction. We conclude that X is completely Hausdorff-closed.

THEOREM 4.2. *A first countable completely Hausdorff space X is first countable and completely Hausdorff-closed if each sequence in X f -accumulates to some point $p \in X$.*

PROOF. The proof is similar to that of Theorem 4.1.

LEMMA 4.3. *Suppose that X is a completely Hausdorff space with the property that every sequence in X with a unique f -accumulation point is convergent. Then every sequence in X has an f -accumulation point.*

PROOF. Suppose there is a sequence, (x_n) , in X which has no f -accumulation point. Fix $p \in X$ and define a sequence, (z_n) , by $z_n = p$ if n is odd and $z_n = x_{n/2}$ if n is even. Clearly p is the unique f -accumulation point of (z_n) and (z_n) does not converge to p .

THEOREM 4.4. *Let (X, T) be a first countable completely Hausdorff space. Then X is first countable and minimal completely Hausdorff if every sequence in X with a unique f -accumulation point is convergent.*

PROOF. Suppose that $h: (X, T) \rightarrow (Y, \sigma)$ is a bijective continuous mapping onto a first countable completely Hausdorff space (Y, σ) . We need to show that h^{-1} is continuous. Let (y_n) be a sequence in Y converging to $y \in Y$. If $p \in X$ is an f -accumulation point of the sequence $(h^{-1}(y_n))$, then the continuity of h shows that $h(p)$ is an f -accumulation point of (y_n) and, therefore, since Y is completely Hausdorff, $h(p) = y$. Thus $h^{-1}(y)$ is the unique f -accumulation point of $(h^{-1}(y_n))$ which implies, by hypothesis, that $(h^{-1}(y_n))$ converges to $h^{-1}(y)$. This shows that h is continuous. We conclude that (X, T) is first countable and minimal completely Hausdorff.

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