REMARKS ON H. H. SCHAEFER'S CONDITIONS

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ABSTRACT. We present two algebraic conditions, each characterizing a partially ordered linear algebra as an algebra of real-valued functions.

In this note we present two algebraic conditions which characterize a Dedekind σ -complete partially ordered linear algebra (dsc-pola) as an algebra of real-valued functions. Work of this type has been done by DeMarr [2], [3]. These two conditions are motivated by Schaefer's work [5, p. 169]; see also [4, p. 2108]. He used these conditions together to characterize a so-called real spectral operator [5, Definitions 3,4] on a locally convex linear space. Roughly speaking a real spectral operator is like a real-valued function.

A dsc-pola A is a real linear associative algebra which satisfies the following conditions. (1) It is partially ordered so that it is a directed partially ordered linear space and $0 \leq xy$ whenever $0 \leq x, 0 \leq y$. (2) It is Dedekind σ -complete, i.e., for $x_n \in A$ and $x_1 \geq x_2 \geq \cdots \geq 0$ implies $\inf\{x_n\}$ exists. In this paper we assume that A has multiplicative identity $1 \geq 0$. Let $I = \{y \in A: y \geq 1, y^{-1} \geq 0\}$. Define $A_1 = \bigcup_{y \in I} \{x \in A: -y \leq x \leq y\}$. Then the multiplication of the elements in A_1 is commutative, and A_1 behaves much like an algebra of real-valued functions. Of particular importance is that the A_1 is an order closed and order convex subalgebra and an f -ring [1, p. 403]. For details of proofs see [2].

Now we consider Schaefer's two conditions as applied to a dsc-pola A . In each case we will show that $A = A_1$.

THEOREM 1. *Suppose A has the following property: for any $x \in A$, $(1 + x^2)^{-1}$ exists and $(1 + x^2)^{-1} \leq 1$. Then $A = A_1$.*

PROOF. First we show that if $y \geq 1$ and y^{-1} exists, then $y \in A_1$. By assumption $(1 + y^{-2})^{-1}$ exists and $y^{-2}(1 + y^{-2})^{-1} = 1 - (1 + y^{-2})^{-1} \geq 0$; therefore, $(1 + y^{-2})^{-1} \geq 0$, so $(1 + y^{-2})^{-1} \in A_1$. This implies $1 \leq 1 + y^{-2} \in A_1$ [2, Theorem I.6.10] or $y^{-2} \geq 0$. Hence, by definition of A_1 we know $y^2 \in A_1$. Since $1 \leq y \leq y^2$ and A_1 is order convex we get $y \in A_1$. Now for any $1 \leq x \in A$, clearly $1 \leq x \leq 1 + x^2$. By assumption $(1 + x^2)^{-1}$ exists and from the above paragraph we get $1 + x^2 \in A_1$. Since A_1 is order convex we get $x \in A_1$. Since A is directed we get $A = A_1$. $\square$

REMARK. The above theorem is true if we consider the following weaker

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property: for any $x \in A$ there exists a strictly positive integer m such that $(1 + x^m)^{-1}$ exists and $(1 + x^m)^{-1} \leq 1$.

THEOREM 2. *Suppose A has the following property: for any $x \in A$, $(1 + x^2)^{-1}$ exists and $x(1 + x^2)^{-1} \leq 1$. Then $A = A_1$.*

PROOF. Replacing x by $-x$ in the assumption we see easily that $-1 \leq x(1 + x^2)^{-1} \leq 1$. Thus for any $x \in A$ $x(1 + x^2)^{-1} \in A_1$. Take any $1 \leq z \in A$. Clearly $1 \leq z \leq 1 + z^2 = y$ and y^{-1} exists. Therefore for all positive integers n , $-1 \leq (y/n)(1 + (y/n)^2)^{-1} \leq 1$. Hence $y^{-1}(1 + (y/n)^2) = y^{-1} + y/n^2 \in A_1$. Since A_1 is closed (order) we have $y^{-1} = \lim_{n \rightarrow \infty} (y^{-1} + y/n^2) \in A_1$, so that $y \in A_1$. This means $z \in A_1$. Since A is directed we have $A = A_1$. $\square$

REMARKS. (1) It can be easily shown that the conditions mentioned above are also necessary conditions. Hence the conditions are equivalent.

(2) Since A_1 itself is a Dedekind σ -complete lattice and since A_1 is an f -ring we can easily verify that 1 is a weak unit, i.e., for $z \geq 0$, $I \wedge z = 0$ imply $z = 0$. Thus for any $x \in A_1$ by Freudenthal theorem [1, p. 364] we know $x = \int_{-\infty}^{\infty} \lambda \, d e(\lambda)$. Where λ is real, $1 \geq e(\lambda)^2 = e(\lambda) \geq 0$ and the integral is defined to be $o - \lim_n \sum_{k=-n}^n (k/n)(e((k+1)/n) - e(k/n))$. This means every $x \in A_1$ is a spectral element in the sense of Schaefer [5, Definitions 3,4].

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