PL INVOLUTIONS ON THE NONORIENTABLE 2-SPHERE BUNDLE OVER S^1

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ABSTRACT. We show that there are exactly nine distinct PL involutions on the nonorientable 2-sphere bundle over S^1 , up to PL equivalences. This, together with results of [1], [3] and [8], classifies all PL involutions on the 2-sphere bundles over S^1 .

1. Introduction. All PL involutions on $S^2 \times S^1$ have been classified [1], [3], [8]. This note is concerned with a complete classification of the PL involutions on the 2-sphere bundles over S^1 . Let M be a closed, nonirreducible 3-manifold and T be a PL involution on M with fixed-point set $\text{Fix}(T)$. It is known [2], [8] that there exists a 2-sphere S in M not bounding a 3-cell such that either $T(S) \cap S = \emptyset$ or $T(S) = S$ and S is in general position with respect to $\text{Fix}(T)$. Based on this, we prove the following.

THEOREM. *The nonorientable 2-sphere bundle N over S^1 admits exactly nine distinct PL involutions, up to PL equivalences.*

In fact, a PL involution on N is PL equivalent to one of the nine standard involutions which we shall describe in the following. We regard S^2 as the unit sphere $\{(x, y, z) | x^2 + y^2 + z^2 = 1\}$ in R^3 . We define some involutions k_i ($i = 1, \dots, 5$) on S^2 as the following:

$$\begin{aligned} k_1(x, y, z) &= (x, y, -z), & k_2(x, y, z) &= (x, -y, -z), \\ k_3(x, y, z) &= (x, -y, z), & k_4(x, y, z) &= (-x, -y, z), \end{aligned}$$

and

$$k_5(x, y, z) = (-x, -y, -z).$$

Let I be the unit interval $[0, 1]$. N may be regarded as the identification space obtained from $S^2 \times I$ by identifying $(x, 0)$ with $(k_1(x), 1)$. If $(x, t) \in S^2 \times I$, we shall denote by $[x, t]$ the image of (x, t) under the identification map. The following is the list of the eight standard involutions h_i ($i = 1, \dots, 8$) on N with fixed-point set F_i .

- (1) $h_1([x, t]) = [k_5(x), t], \quad F_1 = \emptyset;$
- (2) $h_2([x, t]) = [k_4(x), 1 - t], \quad F_2 = S^0;$

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- (3) $h_3([x, t]) = [k_2(x), 1 - t], \quad F_3 = S^1 \dot{\cup} S^0;$
 (4) $h_4([x, t]) = [k_1(x), 1 - t], \quad F_4 = S^2 \dot{\cup} S^1;$
 (5) $h_5([x, t]) = [k_2(x), t], \quad F_5 = S^1 \dot{\cup} S^1;$
 (6) $h_6([x, t]) = [k_4(x), t], \quad F_6 = S^1;$
 (7) $h_7([x, t]) = [k_1(x), t], \quad F_7 = S^1 \times S^1;$
 (8) $h_8([x, t]) = [k_3(x), t], \quad F_8 = K \text{ (Klein bottle).}$

For the last one, N is viewed as the identification space obtained from $S^2 \times I$ by identifying $(x, 0)$ with $(k_5(x), 1)$.

- (9) $h_9([x, t]) = [k_5(x), t], \quad F_9 = S^2.$

In what follows all spaces and maps will be in the PL category.

2. Proof of the Theorem. Let h be an involution on N with fixed-point set F . Since N is not irreducible, there exists a 2-sphere S in N not bounding a 3-cell such that either $h(S) \cap S = \emptyset$ or $h(S) = S$ and S is in general position with respect to F . If $F = \emptyset$, h is equivalent to h_1 (see [7]). Therefore, in what follows we assume that $F \neq \emptyset$. We divide the proof into two cases.

Case 1. $h(S) \cap S = \emptyset$. Split N along S and $h(S)$. Then we have two components N_1 and N_2 , each of which is homeomorphic to $S^2 \times I$. Since $F \neq \emptyset$, N_i ($i = 1, 2$) is invariant under h . Parametrize N_1 and N_2 in terms of $S^2 \times I$ such that $S^2 \times 1 \approx h(S)$ and $S^2 \times 0 \approx S$. Let $h_i = h|_{N_i}$. Then there exists an isotopy of $X^2 \times I$, keeping $S^2 \times 1$ constant, after which a product structure $S^2 \times I$ is defined so that $h_i(x, t) = (g_i(x), 1 - t)$ for $(x, t) \in S^2 \times I$, where g_i is a map of S^2 (with $g_i^2 = \text{identity}$) (see [4], [5], [9]). Therefore, since h is an involution, N may be viewed as the identification space $S^2 \times I/g_2g_1$ (identify $(x, 0)$ with $(g_2g_1(x), 1)$). Thus, we may assume that g_1 is orientation-preserving and g_2 is orientation-reversing. Therefore, F is homeomorphic to one of the following: (1) S^0 , (2) $S^0 \dot{\cup} S^1$, (3) S^2 , and (4) $S^2 \dot{\cup} S^1$. Since there exists an invariant 2-sphere $S_i \subset N_i (\approx S^2 \times \{1/2\})$ after parametrization such that h_i interchanges the sides of S_i , it is not difficult to show that h is equivalent to one of h_i 's ($i = 1, 2, 3, 4, 9$).

Case 2. $h(S) = S$ and S is in general position with respect to F . Let N' be the space obtained by splitting N along S . Then $N' \approx S^2 \times I$, and N may be obtained from $S^2 \times I$ by identifying $(x, 0)$ with $(\phi(x), 1)$, where ϕ is a map of $S^2 (\approx S)$. Let h' be the obvious involution on N' induced by h . Then since S is in general position with respect to F , there exist a product structure $S^2 \times I$ and an involution g on S^2 such that $h'(x, t) = (g(x), t)$ for $(x, t) \in S^2 \times I$ (see [6], [9]). Therefore F is homeomorphic to one of the following: (1) S^1 , (2) $S^1 \dot{\cup} S^1$, (3) $S^1 \times S^1$, (4) K .

Subcase 1. $F \approx S^1$. Let f be the orbit map of h and $f' = f|_S$. Let ϕ' be the map of $S^2 (\approx f'(S))$ induced by ϕ such that $\phi'f' = f'\phi$. Then the orbit space is viewed as $S^2 \times I/\phi'$. Notice that ϕ' is orientation-reversing, and $S^2 \times I/\phi' \approx N$. Now consider the standard involution h_6 on $S^2 \times I/k_1$. Let q be the orbit map of h_6 and $q' = q|_{[S^2 \times \{0\}]}$. Then the orbit space is viewed as $S^2 \times I/k'_1$,

where k'_1 is the map of $S^2 (\approx q'([S^2 \times \{0\}]))$ induced by k_1 such that $q'k_1 = k'_1q'$. Let $a = (0, 0, 1) \in S^2$. Then we may assume that $f(F)$ (and $q(F_6)$) is obtained from $\{a\} \times I \cup \{-a\} \times I \subset S^2 \times I$ by identifying $(a, 0)$ and $(-a, 0)$ with $(-a, 1)$ and $(a, 1)$, respectively (see the product structure of N' above). We shall proceed to define a homeomorphism t of $S^2 \times I/\phi'$ to $S^2 \times I/k'_1$ such that $tf(F) = q(F_6)$. Define a homeomorphism t' of $S^2 \times \{0, 1\} \cup \{a, -a\} \times I \subset S^2 \times I$ by

$$t' = \text{the identity on } S^2 \times \{0\} \cup \{a, -a\} \times I,$$

$$t' = k'_1\phi'^{-1} \text{ on } S^2 \times \{1\}.$$

Then $t'|S^2 \times \{1\}$ is an orientation-preserving homeomorphism, keeping $(a, 1)$ and $(-a, 1)$ fixed. Therefore there exists an isotopy between $t'|S^2 \times \{1\}$ and the identity on $S^2 \times \{1\}$, constant on the fiber over $\{(a, 1), (-a, 1)\}$. Hence there exists an extension $\bar{t}$ of t' to $S^2 \times I$ such that $\bar{t}|[a, -a] \times I$ is the identity. Since $k'_1\bar{t} = \bar{t}\phi'$ on $S^2 \times \{0\}$, $\bar{t}$ induces a homeomorphism t of $S^2 \times I/\phi'$ to $S^2 \times I/k'_1$ such that $tf(F) = q(F_6)$.

Let U' be the interior of a regular neighborhood of $f(\text{Fix}(h))$ in $S^2 \times I/\phi'$. Let $U = S^2 \times I/\phi' - U'$ and $V = S^2 \times I/k' - t(U')$, $U_1 = S^2 \times I/\phi - f^{-1}(U')$ and $V_1 = S^2 \times I/k_1 - q^{-1}(t(U'))$. Then each of U , V , U_1 and V_1 are homeomorphic to the nonorientable annulus bundle over S^1 with connected boundary, and their fundamental groups are isomorphic to that of $S^1 \times S^1$. Let $\bar{f} = f|U$ and $\bar{q} = q|V$. Considering the product structure of N' above and the way that t is defined, one can easily see that $(t\bar{f})_\#(\Pi_1(U_1)) = \bar{q}_\#(\Pi_1(V_1))$. Therefore, there exists a homeomorphism $\hat{t}$ of $S^2 \times I/\phi - F$ to $S^2 \times I/k_1 - F_6$ such that $\hat{q}\hat{t} = t\bar{f}$ on $S^2 \times I/\phi - F$. This $\hat{t}$ can be extended to $S^2 \times I/\phi$ such that $\hat{q}\hat{t} = t\bar{f}$ on $S^2 \times I/\phi$. Therefore h and h_6 are equivalent.

Subcase 2. $F \approx S^1 \cup S^1$. Let M and M_1 be the orbit space of h and h_5 . Then it is easy to see that M and M_1 are homeomorphic to N . One may easily modify the subcase 1 above to see that there exists a homeomorphism t of M to M_1 such that $tf(F) = q(F_5)$, where f and q are the orbit maps of h and h_5 , respectively. We define U , V , U_1 and V_1 in a similar way to those in the subcase 1 above. Then each of U , V , U_1 and V_1 is homeomorphic to the nonorientable annulus bundle over S^1 with disconnected boundary, and their fundamental groups are isomorphic to that of K . Since $\Pi_1(K)$ has only one subgroup of index 2 isomorphic to $\Pi_1(K)$, it follows from the lifting theorem that $h|(N - F)$ and $h_5|(N - F_5)$ are equivalent, and therefore h and h_5 are equivalent in the usual way.

Subcase 3. $F \approx S^1 \times S^1$. Let $c = \{(x, y, 0) \in S^2 | x^2 + y^2 = 1\}$. We may assume that $\text{Fix}(h') = c \times I \subset S^2 \times I$ (see the product structure of N' above). Let $A^+ = \{(x, y, z) \in S^2 | z \geq 0\}$ and $A^- = \{(x, y, z) \in S^2 | z \leq 0\}$. Notice that $\phi(c) = c$ and ϕ interchanges A^+ and A^- . For, otherwise, $\text{Fix}(h) \approx K$ since ϕ is orientation-reversing and $\phi(A^+) = A^+$. Therefore the orbit space $\bar{N}$ of h is essentially obtained from $A^+ \times I$ by identifying $(x, 0)$ with $(h'\phi(x), 1)$. Since $h'\phi$ is orientation-preserving, and therefore it is isotopic to the identity, $\bar{N}$ is a solid torus whose boundary is $f(F)$, where f is the orbit map of h . Thus it follows that h is equivalent to h_7 in the usual way.

Subcase 4. $F \approx K$. We define A^+ , A^- and f as in Subcase 3 above. Since ϕ

is orientation-reversing and $F = K$, $\phi(A^+) = A^+$. Therefore, the orbit space of h is a nonorientable disk bundle over S^1 whose boundary is $f(F)$. Hence it follows that h is equivalent to h_8 .

Thus we now conclude that h is equivalent to one of the nine standard involutions on N as listed in the introduction.

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